

$(R'_2)_{n+1}$ is defined by

$$\frac{I_{2m} + \epsilon_x + (R'_2)_{n+1}}{(R_2)_n - \epsilon_n} = \frac{I_{1m} + \epsilon_x}{(R_1)_n - \epsilon_n}.$$

Since $(R'_2)_{n+1}$ tends to $(R_2^*)_{n+1}$ if ϵ_x and ϵ_n tend to zero, it follows that the rate of $C_i^{(n)}$ tends to R_n defined in (11).

We split the information into p pairs of $\{ \lfloor p(\bar{B} + 2\epsilon) \rfloor, \lfloor p(\bar{H} + 2\epsilon) \rfloor \}$ bit and $a_p - D$ pairs of $\{ 0, \lfloor p(q_m - \delta)(R'_2)_{n+1} \rfloor \}$ bit. The first part is transmitted in packets \mathcal{P}_i , with the Tolhuizen scheme. In \mathcal{P}_i , $D < i \leq a_p$, we encode the m states with $\{ \lfloor (I_{1m} + \epsilon_x)(q_m + \delta)p \rfloor, \lfloor (I_{2m} + \epsilon_x)(q_m + \delta)p \rfloor \}$ symbols. User 2 adds $\lfloor p(q_m - \delta)(R'_2)_{n+1} \rfloor$ new information symbols to these and fills up until he has $\lfloor (I_{2m} + \epsilon_x + (R'_2)_{n+1})(q_m + \delta)p \rfloor$. All of them are transmitted together, using the (asymmetrical) code $C_i^{(n)}$ with

$$i' := \left\lfloor \frac{(I_{1m} + \epsilon_x)(q_m + \delta)p}{(R_1)_n - \epsilon_n} \right\rfloor$$

$$= \left\lfloor \frac{(I_{2m} + \epsilon_x + (R'_2)_{n+1})(q_m + \delta)p}{(R_2)_n - \epsilon_n} \right\rfloor.$$

Now the proof can be finished in the same way as in Theorem 5.

One final remark concerns the convergence of the sequence R_n . In the symmetrical case the points (R_n, R_n) are all on the line $R_1 = R_2$, and their distance to the origin increases monotonically. In the asymmetrical case, however, the points $((R_1)_n, (R_2)_n)$ are not on a straight line. Here we must show that the sequence $(\eta_n)_{n \in \mathbb{N}}$ is monotonically decreasing (or increasing, depending on $\eta_1 > 1$ or $\eta_1 < 1$); hence $(R_1)_n$ is increasing (decreasing) and has a limit R_1 . Although $(R_2)_n$ is not monotonic, it can be shown that this sequence converges, too. We find that the limit (R_1, R_2) satisfies (1), which is what we had to show.

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A New Geometric Capacity Characterization of a Discrete Memoryless Channel

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Abstract—A novel geometrical characterization of capacity of a discrete memoryless channel is proposed according to Csiszár's theorem, which represents the capacity using the Kullback–Leibler discrimination information. As a result, a new geometrical capacity computing method is given.

I. INTRODUCTION

Computational methods for the capacity of a discrete memoryless channel proposed to date may be divided into direct computing methods [1]–[3] and sequential computing methods [4]–[6]. In the direct computing method the capacity C is calculated according to linear equation theory and the Kuhn–Tucker condition of convex programming. In the sequential computing method a sequence $\{p^n\}_{n=0}^\infty$ starting at an appropriate initial probability distribution p^0 is defined, and it is shown that the sequence converges to a probability distribution attaining C . Furthermore, the convergence speed is estimated.

The present correspondence belongs to the direct computing category. Since Muroga's method [1] is based on the linear equation theory, it is not easy to understand the relation between the row probability vectors of a channel matrix and the probability vector attaining C . After characterizing C geometrically, we present a new computational method based on Csiszár's theorem which describes C as the solution of a minimax problem using the Kullback–Leibler discrimination information.

II. DEFINITIONS

Let $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ be the input and output alphabets, respectively. Let $Q(y_j|x_i)$ be the conditional probability of y_j when x_i is given. We treat a discrete memoryless channel whose channel matrix is

$$Q = (Q(y_j|x_i)), \quad i=1, \dots, m, j=1, \dots, n.$$

(The (i, j) entry of Q is $Q(y_j|x_i)$.) If there is no ambiguity, we also call Q itself a channel. Let

$$Q^i = (Q(y_1|x_i), \dots, Q(y_n|x_i))$$

be the probability distribution on Y when x_i is transmitted. We denote a probability distribution on X by $p(x_i)$ and that on Y corresponding to $p(x_i)$ by $q(y_j)$; i.e.,

$$q(y_j) = \sum_{i=1}^m p(x_i) Q(y_j|x_i), \quad \text{or } q = pQ.$$

We define two sets of probability distributions:

$$\Delta^n = \left\{ (\alpha_1, \dots, \alpha_n) \left| \sum_{j=1}^n \alpha_j = 1, \alpha_j > 0 (j=1, \dots, n) \right. \right\}$$

$$\bar{\Delta}^n = \left\{ (\alpha_1, \dots, \alpha_n) \left| \sum_{j=1}^n \alpha_j = 1, \alpha_j \geq 0 (j=1, \dots, n) \right. \right\}.$$

We define the Kullback–Leibler information by

$$D(q^1||q^2) \triangleq \sum_{j=1}^n q_j^1 \log(q_j^1/q_j^2)$$

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where $q^1 = (q_1^1, \dots, q_n^1)$, $q^2 = (q_1^2, \dots, q_n^2) \in \bar{\Delta}^n$. It is well-known that $D(q^1 \| q^2)$ has the following properties:

- 1) $D(q^1 \| q^2) \geq 0$: equality holds if and only if $q^1 = q^2$;
- 2) in general, neither $D(q^1 \| q^2) = D(q^2 \| q^1)$ nor $D(q^1 \| q^2) + D(q^2 \| q^3) \geq D(q^1 \| q^3)$ holds;
- 3) convexity: if $q = (1 - \lambda)q^1 + \lambda q^2$ and $\bar{q} = (1 - \lambda)\bar{q}^1 + \lambda \bar{q}^2$ ($0 \leq \lambda \leq 1$), then $D(q \| \bar{q}) \leq (1 - \lambda)D(q^1 \| \bar{q}^1) + \lambda D(q^2 \| \bar{q}^2)$.

For k points q^1, \dots, q^k in \mathbb{R}^n , let $C(q^1, \dots, q^k)$ be their convex hull and $q^1 \cup \dots \cup q^k$ be the minimum linear subspace they span.

III. LEMMAS AND KNOWN THEOREMS

Lemma 1: Any $q \in C(q^1, \dots, q^k)$ is contained in a simplex with vertices belonging to $\{q^1, \dots, q^k\}$ (see [7, p. 15]).

Lemma 2: If three distinct points $q^1, q^2, q^3 \in \Delta^n$ are located on a line in this order, the following inequality holds:

$$D(q^1 \| q^2) + D(q^2 \| q^3) < D(q^1 \| q^3).$$

Proof: From this condition, a positive number α exists such that $q^1 - q^2 = \alpha(q^2 - q^3)$. Thus if $q^i = (q_1^i, \dots, q_n^i)$ ($i = 1, 2, 3$) we have

$$\begin{aligned} & D(q^1 \| q^3) - D(q^1 \| q^2) - D(q^2 \| q^3) \\ &= \sum_{j=1}^n (q_j^1 - q_j^2) \log(q_j^2 / q_j^3) \\ &= \alpha \sum_{j=1}^n (q_j^2 - q_j^3) \log(q_j^2 / q_j^3) \\ &= \alpha \{ D(q^2 \| q^3) + D(q^3 \| q^2) \} > 0. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 3: For $q^1, q^2 \in \Delta^n$ ($q^1 \neq q^2$), $\lambda \geq 0$,

$$f(\lambda) = D(q^1 \| (1 - \lambda)q^1 + \lambda q^2)$$

and

$$g(\lambda) = D((1 - \lambda)q^1 + \lambda q^2 \| q^1)$$

are both increasing functions of λ .

Proof: Lemma 3 is trivial by Lemma 2. Q.E.D.

Lemma 4: Let V be a closed convex subset of $\bar{\Delta}^n$. For $q \in \bar{\Delta}^n$, if there is some $r \in V$ such that $D(r \| q) < \infty$, then a unique $r^0 \in V$ exists minimizing

$$D(r \| q), \quad (r \in V)$$

(see [8, p. 59]).

We call this r^0 the projection of q onto V , and denote it by

$$r^0 = \text{pr}_V(q).$$

We define a linear set E in $\bar{\Delta}^n$ by

$$E = \left\{ q = (q_1, \dots, q_n) \in \bar{\Delta}^n \mid \sum_{j=1}^n a_{kj} q_j = b_k, \right.$$

a_{kj}, b_k are constants and k ranges over a finite index set $\left. \right\}$.

For example, a straight line connecting two probability distributions in $\bar{\Delta}^n$, a plane determined by three distributions, and so forth, are linear sets.

Pythagoras Theorem: Let E be a linear set in $\bar{\Delta}^n$. For $r \in \bar{\Delta}^n$, let $q = \text{pr}_E(r)$. Then for any $s \in E$,

$$D(s \| q) + D(q \| r) = D(s \| r)$$

holds (see [8, p. 59]).

Theorem (Csiszár): The capacity of a discrete memoryless channel Q is equal to

$$C = \min_{q \in \bar{\Delta}^n} \max_{1 \leq i \leq m} D(Q^i \| q).$$

Furthermore, $q^0 \in \bar{\Delta}^n$ which achieves the minimum is unique and $q^0 = p^0 Q$, where p^0 is any probability distribution that maximizes the mutual information $I(p, Q)$ (see [8, p. 142, 147]).

Theorem (Kuhn-Tucker): An input probability distribution p maximizes $I(p, Q)$ if and only if a constant C exists satisfying

$$D(Q^i \| pQ) \begin{cases} = C, & \text{if } p(x_i) > 0 \\ \leq C, & \text{if } p(x_i) = 0 \end{cases}$$

(see [9, p. 91]).

IV. GEOMETRIC CHARACTERIZATION OF CAPACITY

Without loss of generality we let Q^1, \dots, Q^k be the extreme points of $V = C(Q^1, \dots, Q^m)$.

Theorem 1: If a probability distribution q^0 satisfying

$$D(Q^i \| q^0) = C \text{ (constant for } i = 1, \dots, k)$$

is in V , C is the capacity of the channel Q .

Proof: From Lemma 1, a subset of $\{Q^1, \dots, Q^k\}$ exists, say, $\{Q^1, \dots, Q^r\}$, such that

$$\begin{aligned} q^0 &= \sum_{i=1}^r \alpha_i Q^i \\ \sum_{i=1}^r \alpha_i &= 1, \alpha_i > 0, i = 1, \dots, r. \end{aligned}$$

Defining an input probability distribution p^0 by

$$p^0(x_i) = \begin{cases} \alpha_i, & i = 1, \dots, r \\ 0, & i = r+1, \dots, k, \end{cases}$$

we obtain $q^0 = p^0 Q$. From the theorem assumption, for $i = 1, \dots, r$, we have

$$D(Q^i \| p^0 Q) = D(Q^i \| q^0) = C.$$

On the other hand, for $i = r+1, \dots, m$, from Lemma 1 a subset of $\{Q^1, \dots, Q^k\}$ exists, say, $\{Q^1, \dots, Q^s\}$, such that

$$\begin{aligned} q^i &= \sum_{h=1}^s \beta_h Q^h \\ \sum_{h=1}^s \beta_h &= 1, \beta_h > 0, \quad h = 1, \dots, s. \end{aligned}$$

Therefore, using the convexity of D , we have

$$\begin{aligned} D(Q^i \| q^0) &= D\left(\sum_{h=1}^s \beta_h Q^h \| q^0 \right) \\ &\leq \sum_{h=1}^s \beta_h D(Q^h \| q^0) \\ &= C, \quad i = r+1, \dots, m. \end{aligned}$$

Consequently, p^0 satisfies the Kuhn-Tucker condition, and q^0 attains the capacity. These results are independent of the choice of points representing q^0 and Q^i as convex linear combinations. Q.E.D.

A distribution $q \in \bar{\Delta}^n$ satisfying

$$D(Q^1 \| q) = \dots = D(Q^k \| q)$$

is called an equidistant point from Q^1, \dots, Q^k . According to the previous theorem, we find that when we try to compute C , it is

not necessary to consider the probability vectors that are not the extreme points of V .

Even if the point equidistant from the extreme points of V exists, it may not be in V . Since the unique q^0 that attains the capacity must be in V , any equidistant point outside V does not attain it. The following theorem specifies the relation between an equidistant point $q \notin V$ and the capacity-achieving point q^0 .

Theorem 2: If q is equidistant from Q^1, \dots, Q^k , then $q^0 = \text{pr}_{\cdot V}(q)$ achieves the capacity.

Proof: By Lemma 1 a subset of $\{Q^1, \dots, Q^k\}$ exists, say, $\{Q^1, \dots, Q^t\}$, such that

$$q^0 = \sum_{i=1}^t \gamma_i Q^i$$

$$\sum_{i=1}^t \gamma_i = 1, \quad \gamma_i > 0, \quad i=1, \dots, t.$$

Thus denoting $E = Q^1 \cup \dots \cup Q^t$ we have

$$q^0 = \text{pr}_{\cdot V}(q) = \text{pr}_{\cdot E}(q).$$

The Pythagoras theorem shows that

$$D(Q^i \| q^0) = C \quad (\text{constant for } i=1, \dots, t).$$

If we can show

$$D(Q^i \| q^0) \leq C, \quad i=t+1, \dots, k,$$

the proof is completed. Let L_i be the line connecting Q^i ($i=t+1, \dots, k$) and q^0 . First, we show that on the line L_i , $q^i = \text{pr}_{\cdot L_i}(q)$, q^0 , and Q^i are located in this order. Suppose q^i is between q^0 and Q^i . Since the line segment connecting q^0 and Q^i is included in V , we have

$$D(q^0 \| q) \leq D(q^i \| q) \quad (1)$$

because of the minimality of $D(q^0 \| q)$. On the other hand, by the Pythagoras theorem,

$$D(q^0 \| q^i) + D(q^i \| q) = D(q^0 \| q)$$

holds, and therefore,

$$D(q^i \| q) < D(q^0 \| q).$$

However, this contradicts (1).

Next, suppose q^0 , Q^i , and q^i are located in this order. By the Pythagoras theorem and Lemma 3, we have

$$\begin{aligned} D(q^0 \| q) &= D(q^0 \| q^i) + D(q^i \| q) \\ &> D(Q^i \| q^i) + D(q^i \| q) \\ &= D(Q^i \| q). \end{aligned}$$

This also contradicts the minimality of $D(q^0 \| q)$. Therefore, it has been shown that q^i , q^0 , and Q^i are in this order on the line L_i . Now when $i=t+1, \dots, k$, by the theorem assumption, we have

$$D(Q^1 \| q) = D(Q^i \| q). \quad (2)$$

Furthermore, by the Pythagoras theorem, we have

$$D(Q^1 \| q^0) + D(q^0 \| q) = D(Q^1 \| q) \quad (3)$$

$$D(Q^i \| q^i) + D(q^i \| q) = D(Q^i \| q) \quad (4)$$

$$D(q^0 \| q^i) + D(q^i \| q) = D(q^0 \| q). \quad (5)$$

Therefore, from Lemma 2 and (2)–(5) we have

$$\begin{aligned} D(Q^1 \| q^0) &\leq D(Q^i \| q^i) - D(q^0 \| q^i) \\ &= D(Q^i \| q) - D(q^0 \| q) \\ &= D(Q^1 \| q) - D(q^0 \| q) \\ &= D(Q^1 \| q^0) \\ &= C. \end{aligned} \quad \text{Q.E.D.}$$

From now on, we assume that

$$\dim(Q^1 \cup \dots \cup Q^k) = k-1,$$

i.e., k points Q^1, \dots, Q^k are in the general position. In this case, the point q equidistant from Q^1, \dots, Q^k always exists and it is represented as

$$q = \sum_{i=1}^k \lambda_i Q^i.$$

If $\lambda_i \geq 0$ for all $i=1, \dots, k$, the $q \in V$, and so q achieves the capacity by Theorem 1. Muroga [1] indicates that "if $\lambda_i < 0$ for at least one i , choose $k-1$ points arbitrarily from Q^1, \dots, Q^k and represent the point equidistant from these $k-1$ points as a linear combination shown above. Repeat this calculation for all possible choices of $k-1$ points. If there exist cases where all the coefficients are nonnegative, the maximum transmission rate among them is the capacity. Otherwise, reduce the number of points to $k-2$, $k-3, \dots$, and do a similar calculation until we have some cases where all the coefficients are nonnegative."

This method is correct but contains much redundancy. A more effective method is proposed below.

Theorem 3: We can obtain the q^0 which achieves the capacity by a maximum of $k-2$ projections onto linear sets.

Since $q^0 = \text{pr}_{\cdot V}(q)$ according to Theorem 2, in principle we can obtain q^0 by one projection. However, it is difficult to calculate q^0 using this method. In fact, when we want to project q onto V , we must solve the minimum problem

$$\min_{r \in V} D(r \| q).$$

However, in general, q^0 is on the boundary of V , so we cannot use Lagrange's method of indeterminate coefficients to solve it. Theorem 3 offers an algorithmic method to obtain q^0 which circumvents the difficulty at the sacrifice of a possible increase in number of iterations. Here "algorithmic" means the iterative projections onto linear sets, in which case we can use Lagrange's method.

Proof: Represent the point q equidistant from the extreme points of V as

$$q = \sum_{i=1}^k \lambda_i Q^i,$$

$$\sum_{i=1}^k \lambda_i = 1, \quad \lambda_1, \dots, \lambda_{k_1} > 0, \quad \lambda_{k_1+1}, \dots, \lambda_k \leq 0.$$

Denoting $E^1 = Q^1 \cup \dots \cup Q^{k_1}$ and $q^1 = \text{pr}_{\cdot E^1}(q)$, we have

$$D(Q^i \| q^1) \begin{cases} = C_1, & \text{constant for } i=1, \dots, k_1 \\ \leq C_1, & i=k_1+1, \dots, k. \end{cases}$$

In fact, the equality for $i=1, \dots, k_1$ holds by the Pythagoras theorem. For $i=k_1+1, \dots, k$, let L_{i_1} be the line connecting q^1 and Q^i , and let $r^{i_1} = \text{pr}_{\cdot L_{i_1}}(q)$. Then we find, as previously mentioned in proving Theorem 2, that Q^i , q^1 , and r^{i_1} are located on L_{i_1} in this order. Thus we have

$$\begin{aligned} D(Q^i \| q^1) &\leq D(Q^i \| r^{i_1}) \\ &= C_1. \end{aligned}$$

Now suppose q^1 is represented as

$$q^1 = \sum_{i=1}^{k_1} \mu_i Q^i$$

$$\sum_{i=1}^{k_1} \mu_i = 1, \mu_1, \dots, \mu_{k_2} > 0, \mu_{k_2+1}, \dots, \mu_{k_1} \leq 0.$$

Denoting $E^2 = Q^1 \cup \dots \cup Q^{k_2}$ and $q^2 = \text{pr}_{E^2}(q^1)$, we have

$$D(Q^i \| q^2) \begin{cases} = C_2, & \text{constant for } i=1, \dots, k_2 \\ \leq C_2, & i = k_2+1, \dots, k_1 \end{cases}$$

in the same way as before. For $i = k_1+1, \dots, k$, it can be shown that $D(Q^i \| q^2) \leq C_2$ holds as follows. Let L_{2i} be the line connecting q^2 and Q^i ($i = k_1+1, \dots, k$), and let $r^{2i} = \text{pr}_{L_{2i}}(q)$. Then it is evident from the previous argument that on the line L_{2i} , Q^i, q^2, r^{2i} are in this order. Therefore, we have

$$\begin{aligned} D(Q^i \| q^2) &\leq D(Q^i \| r^{2i}) - D(q^2 \| r^{2i}) \\ &= D(Q^i \| q^1) - D(q^2 \| q^1) \\ &\leq C_1 - D(q^2 \| q^1) \\ &= D(Q^1 \| q^2) \\ &= C_2, \quad i = k_1+1, \dots, k. \end{aligned}$$

We iterate this procedure to obtain q^1, q^2, q^3, \dots until all coefficients are positive. Let k_{i+1} be the number of Q^j having positive coefficients in the representation of q^i . The worst case is that $k_{i+1} = k_i - 1$ holds for all $i = 1, 2, \dots$. Since the point equidistant from two points always belongs to the line segment connecting them, we can obtain q^0 that achieves the capacity by a maximum of $k-2$ iterative projections. Q.E.D.

Now we assume

$$\dim(Q^1 \cup \dots \cup Q^k) = d-1 < k-1.$$

In this case, a point equidistant from Q^1, \dots, Q^k does not exist. However, d points chosen arbitrarily from Q^1, \dots, Q^k are in the general position. Therefore, there exists a point equidistant from these d points. Thus by using the foregoing method we obtain ${}_k C_d$ values of channel capacity for all combinations. The following theorem ensures that the maximum among these ${}_k C_d$ values is the true capacity.

Theorem 4: If $\dim(Q^1 \cup \dots \cup Q^k) = d-1$, the maximum value among the ${}_k C_d$ values of "capacity" computed for d points chosen arbitrarily from Q^1, \dots, Q^k is the true capacity.

Proof: Let Q^1, \dots, Q^h be the points in Q^1, \dots, Q^k such that $D(Q^1 \| q^0) = \dots = D(Q^h \| q^0)$ is the greatest value among $D(Q^1 \| q^0), \dots, D(Q^k \| q^0)$, where q^0 is the capacity-achieving point. Since $q^0 \in C(Q^1, \dots, Q^h)$, by Lemma 1 a subset of $\{Q^1, \dots, Q^h\}$ exists, say, $\{Q^1, \dots, Q^{h_1}\}$, such that q^0 is contained in the simplex having Q^1, \dots, Q^{h_1} vertices. Then $h_1 \leq d$, and if we solve Csiszár's minimax problem for any d points including those h_1 points, we obtain the true capacity C . The rates for the other choices of d points are, of course, not greater than C . Q.E.D.

V. EXAMPLES

1) 2×2 Channel Matrix: The capacity C of a channel

$$Q = \begin{pmatrix} Q^1 \\ Q^2 \end{pmatrix} = \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}, \quad 0 \leq a, b \leq 1,$$

is

$$C = \begin{cases} \log(1+e^A) - (1-b)H^1/(a-b) \\ \quad + (1-a)H^2/(a-b), & a \neq b \\ 0, & a = b, \end{cases}$$

where

$$H^1 = -a \log a - (1-a) \log(1-a)$$

$$H^2 = -b \log b - (1-b) \log(1-b)$$

and

$$A = (H^1 - H^2)/(a-b).$$

In this case, the convex hull V of Q^1, Q^2 is the line segment connecting Q^1 and Q^2 . Since the equidistant point q^0 from Q^1 and Q^2 always exists in V , $C = D(Q^1 \| q^0)$ is the capacity by Theorem 1.

2) 3×3 Channel Matrix: Next, we consider a channel

$$Q = \begin{pmatrix} Q^1 \\ Q^2 \\ Q^3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\left(\sum_{i=1}^3 a_i = \sum_{i=1}^3 b_i = \sum_{i=1}^3 c_i = 1, a_i, b_i, c_i \geq 0, i=1, 2, 3 \right).$$

For two probability distributions Q^i, Q^j ($i \neq j$), the "midpoint" of Q^i and Q^j is defined by a point M^{ij} which satisfies the following:

- 1) M^{ij} is on the line segment connecting Q^i and Q^j ;
- 2) $D(Q^i \| M^{ij}) = D(Q^j \| M^{ij})$.

Further, we call $D(Q^i \| M^{ij})$ the "half-length" of the line segment connecting Q^i and Q^j and denote it by $d(Q^i, Q^j)$. By definition, we have $d(Q^i, Q^j) = d(Q^j, Q^i)$. Without loss of generality, we may assume that $d(Q^1, Q^2)$ is the greatest value among $d(Q^1, Q^2)$, $d(Q^2, Q^3)$, and $d(Q^3, Q^1)$. Let q^0 be the equidistant point from Q^1, Q^2, Q^3 ; i.e., q^0 is the unique solution of the following equation:

$$D(Q^1 \| q^0) = D(Q^2 \| q^0) = D(Q^3 \| q^0).$$

Then we have

$$C = \begin{cases} D(Q^1 \| q^0), & \text{if } D(Q^3 \| M^{12}) > D(Q^1 \| M^{12}) \\ D(Q^1 \| M^{12}), & \text{if } D(Q^3 \| M^{12}) \leq D(Q^1 \| M^{12}). \end{cases}$$

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