LETTER

G/D/1 Queueing Analysis by Discrete Time Modeling

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SUMMARY $G/D/1$ is a theoretic model for ATM network queueing based on processing cells. We investigate the $G/D/1$ system by discrete time modeling. Takács' combinatorial methods are applied to analyze the system performance. An approximation for the survivor function $P[Q > q]$, which is the probability that the queue length Q in the stationary state exceeds q, is obtained. The obtained formula requires only very small computational complexity and gives good approximation for the true value of $P[Q > q]$.
key words: *queueing, discrete time approximation, $G/D/1$, ATM*

1. Introduction

In the ATM network, since a cell is the unit of transmission and exchanging, the processing time for one cell is commonly used as the unit of time. Therefore, in order to evaluate system performance, for example, cell congestion, cell transmission delay, and so on, the queueing theoretic model $G/D/1$ is investigated. Among the models belonging in $G/D/1$, $M/D/1$ model is especially the most fundamental and important one. Various types of cell traffic models based on bursty nature of cell streams are proposed and studied, however, many of them are elaborate to handle. Furthermore, the statistical nature of future cell traffic is unknown, so it cannot be determined at this point of time. Hence, it is very common to use $M/D/1$ model to analyze the system performance in the real system design [5].

In this letter, we investigate $G/D/1$ queueing by the discrete time modeling to obtain an approximation of the survivor function $P[Q > q]$, that is the probability that the queue length Q in the stationary state exceeds q. We confirm the accuracy of our approximation by comparison with the exact formula or simulation, especially for $M/D/1$, $H_2/D/1$ and $E_2/D/1$ systems.

2. Discrete Time Modeling

Takács [4] investigated the probability distribution of the maximum value of a random variable which takes non-negative integral values using the combinatorial method. We apply the results of Takács to queueing problems to obtain an approximation of $P[Q > q]$ of

$G/D/1$ systems. Takács' theorems have sometimes been applied to queueing problems [1]. Our approach is to apply Takács' theorem on the waiting time probability to obtain an approximation value of $P[Q > q]$.

We study a single server queue with infinite length buffer. The unit of time is the service time for one cell and the time axis is divided into intervals of the unit time length. Each interval is called a time slot.

2.1 Queue Length Distribution

Let ρ denote the average number of arriving cells during one time slot. For the stability of the system $\rho < 1$ should hold. Let Q_t denote the queue length at the beginning of the t th time slot, and a_t the number of arriving cells during the t th time slot. If the system is not empty, one cell is served at the end of the slot. We have the following recursion for a_t and Q_t;

$$Q_{t+1} = \max(0, Q_t - 1 + a_t), \quad Q_0 = 0 \qquad (1)$$

Iterative operations lead to the solution;

$$Q_t = \max_{0 \le i \le t} \sum_{k=i}^{t-1} (a_k - 1), \qquad (2)$$

where the empty sum implies 0. If $\{a_t\}$ is an i.i.d. sequence, then we can derive the following approximation of $P[Q > q]$.

Let Q denote the stationary queue length, i.e., $Q = \lim_{t \to \infty} Q_t$, then for $Q > 0$ we have by (2),

$$Q = \sup_{i \ge 1} (A_i - i), \qquad (3)$$

where A_i denotes the stationary number of arriving cells during consecutive i time slots. By applying Takács' theorems ([4] p.15, Theorem 3, p.17, Theorem 4) to (3), we have

$$P[Q > q] = (1 - \rho) \sum_{i=1}^{\infty} P[A_i = i + q + 1] \qquad (4)$$

$$= 1 - r_{q+1} \qquad (5)$$

where r_q is defined by

$$\frac{(1 - \rho)\zeta(z)}{\zeta(z) - z} = \sum_{q=0}^{\infty} r_q z^q \qquad (6)$$

with $\zeta(z)$ the probability generating function of a_t. The formulas (4) and (5) are exact formulas of $P[Q > q]$, however, they are not suitable for real computation because they require much computational time. We want a more tractable formula (which may not be exact).

2.2 Waiting Time Distribution

Let W_n denote the waiting time of the nth cell, τ_n the inter- arrival time between nth and $(n + 1)$ st cells. We then have the following recursion;

$$W_{n+1} = \max(0, W_n + 1 - \tau_n), \quad W_0 = 0 \qquad (7)$$

The solution of (7) is given by

$$W_n = \max_{0 \le i \le n} \sum_{k=i}^{n-1} (1 - \tau_k), \qquad (8)$$

where we assume that the empty sum implies 0. From now to the end of this paper, we assume that $\{\tau_n\}$ is an i.i.d. sequence. Let W denote the stationary waiting time, i.e., $W = \lim_{n \to \infty} W_n$. For $W > 0$ we have by (8)

$$W = \sup_{i \ge 1} (i - T_i), \qquad (9)$$

where T_i denotes the stationary inter-arrival time between nth and $(n + i)$th cells. By applying Takács' theorems ([4], p.24, Theorem 1, p.25, Theorem 3) to (9), we have

$$P[W > w] = \sum_{j=w+1}^{\infty} \frac{w+1}{j} P[T_j = j - w - 1] \qquad (10)$$

$$= \delta^{w+1} \qquad (11)$$

where δ is defined as the minimum positive root of the eq. $\pi(\delta) = \delta$ with $\pi(z)$ the probability generating function of the cell inter-arrival time.

The formula (11) requires only small computational complexity, but our goal is $P[Q > q]$. We modify the idea leading (11) to obtain an approximation of $P[Q > q]$.

3. Approximation of $P[Q > q]$

The cell loss probability is one of the most important performance measures in the ATM network for the design of network parameters. The cell loss probability of a system with a buffer of size $q + 1$ is bounded above by $P[Q > q]$ of the model with an infinite length buffer. Our goal is to obtain a good approximation of $P[Q > q]$ that requires small computational complexity. The author[3] obtained a good approximation of $P[Q > q]$ of a general $G/G/1$ system which is, of course applicable to $G/D/1$, however, it requires much computational complexity.

Now, we give an approximation of $P[Q > q]$ by applying Takács' theorem. The only assumption we need

is that $\{\tau_n\}$ is i.i.d., where τ_n is the inter-arrival time between nth and $(n + 1)$st cells. Because we assumed in our $G/D/1$ model that the system's unit time equals the service time for one cell, when a cell just arrived the number of cells in the buffer equals the waiting time of that cell. Based on this observation, we here propose an approximation;

$$P[Q > q] \approx \frac{\delta^{q+1}}{\rho} \qquad (12)$$

where ρ is the mean cell arrival rate and δ the minimum positive root of the eq. $\pi(\delta) = \delta$ with $\pi(z)$ the probability generating function of cell inter-arrival time.

3.1 $M/D/1$

The exact formula of $P[Q > q]$ of $M/D/1$ model is given by

$$P[Q > q] = 1 - (1 - \rho) \sum_{j=0}^{q} \frac{(-\rho j)^{q-j}}{(q-j)!} \exp(\rho j) \quad (13)$$

from (4) and Jensen's eq.

$$\sum_{j=0}^{\infty} \frac{(a+jb)^j}{j!} e^{-(a+jb)} = \frac{1}{1-b}. \qquad (14)$$

The formula (13) is not suitable for real computation because (13) includes the alternating addition of positive and negative numbers of large absolute values. A more effective algorithm is the following;

$$P[Q > q] = 1 - \sum_{j=0}^{q} p_j, \qquad (15)$$

where $p_j = P[Q = j]$ and

$$p_0 = 1 - \rho, \quad p_1 = (1 - \rho)(-1 + e^{\rho}),$$

$$p_j = \sum_{k=1}^{j} e^{\rho} \frac{(-\rho)^{k-1}}{(k-1)!} p_{j-k}, \quad j \ge 2. \qquad (16)$$

An approximation of $P[Q > q]$ obtained by the author[3] is

$$P[Q > q] \approx \frac{\rho^2 e^{1-\rho}}{1 - \rho e^{1-\rho}} \sum_{i \ge 1} \left(\frac{\rho i}{i+q}\right)^{i+q} e^{i+q-\rho i}. \quad (17)$$

The above formulas (13), (15) and (17) require much computational complexity. Moreover (15) has a problem of under flow when we compute them by a computer. Each term in the summation of (15) is small for large q and j. So, for any q and j sufficiently large, the terms are regarded as 0 because of the precision limit of the computer. For example, if $\rho = 0.8$, we have the same values of $P[Q > q]$ for any $q \ge 70$.

Our formula (12) applied to $M/D/1$ is $P[Q > q] \approx \delta^{q+1}/\rho$ where $\rho < 1$ is the mean cell arrival rate and δ is the minimum positive root of the eq. $\delta(\rho - \log \delta) = \rho$.

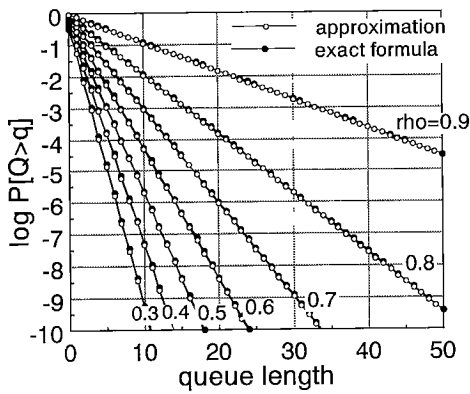We will show in Fig. 1 a comparison of the exact formula and our approximation (12).

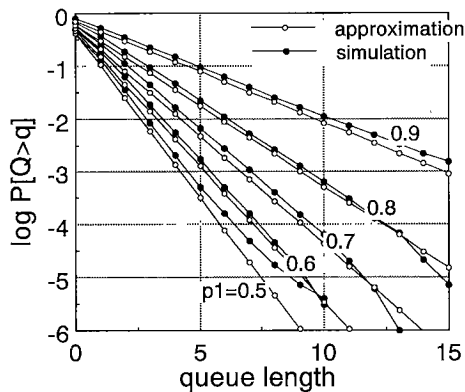**Fig. 1** Approximation of $P[Q>q]$ of $M/D/1$.

Fig. 2 Approximation of $P[Q > q]$ of $H2/D/1$ with rho $1=1.0$, rho $2=0.1$, p1$=0.5$–0.9.

3.2 $H_k/D/1$

H_k denotes the hyperexponential distribution whose probability density function is the mixture of k probability density functions of exponential distributions. Let ρ_i denote the mean of each exponential distribution and p_i the mixture probability, $i = 1, \cdots, k$. Then our approximation becomes δ^{q+1}/ρ where δ is the minimum positive root of the equation.

$$\sum_{i=1}^{k} \frac{p_i \rho_i}{\rho_i - \log \delta} = \delta. \tag{18}$$

We will show in Fig. 2 the comparison with computer simulation.

3.3 $E_k/D/1$

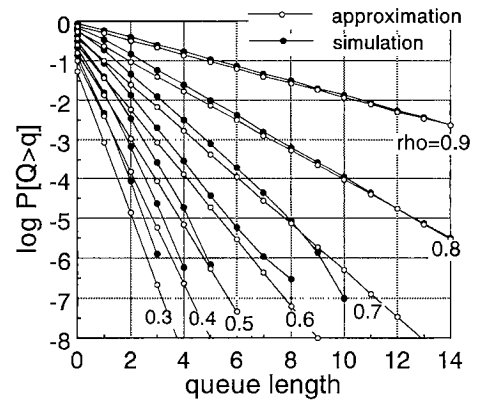E_k denotes the k-Erlang distribution, i.e., it is the distribution of the sum of k random variables that are

Fig. 3 Approximation of $P[Q > q]$ of $E2/D/1$ with rho $=$ 0.3–0.9.

dominated by exponential distributions of mean ρ/k. Our approximation is δ^{q+1}/ρ where δ is the minimum positive root of the equation.

$$\left(\frac{\rho k}{\rho k - \log \delta}\right)^k = \delta. \tag{19}$$

We will show in Fig. 3 the comparison with computer simulation.

4. Conclusion

We proposed in this letter an approximation of the $P[Q > q]$ which requires only small computational complexity. We confirmed the accuracy of the approximation by comparison with the exact formula or simulation in the case of $M/D/1$, $H_2/D/1$ and $E_2/D/1$. The technique derived in this paper is applicable to other $G/D/1$ or $D/G/1$ systems.

Further study is necessary to develop a good approximation of the $P[Q > q]$ of $G/G/1$ systems.

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