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On the Exponential Decay Rate of the Tail of a Discrete Probability Distribution

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ABSTRACT

We give a sufficient condition for the exponential decay of the tail of a discrete probability distribution $\pi = (\pi_n)_{n\geq 0}$ in the sense that $\lim_{n\to\infty}(1/n)\log\sum_{i>n}\pi_i = -\theta$ with $0 < \theta < \infty$. We focus on analytic properties of the probability generating function of a discrete probability distribution, especially, the radius of convergence and the number of poles on the circle of convergence. Furthermore, we give an example of an M/G/1 type Markov chain such that the tail of its stationary distribution does not decay exponentially.

Key Words: Cauchy–Hadamard's theorem; Complex function theory; Exponential decay rate; Markov chain; Tail of distribution.

1. INTRODUCTION

Let us consider a discrete probability distribution $\pi = (\pi_n)_{n\geq 0}$ and define the tail of the π by $\Pi_n \equiv \sum_{i>n}\pi_i$. If Π_n decreases as

$$\lim_{n\to\infty}\frac{1}{n}\log\Pi_n = -\theta, \quad 0 < \theta < \infty, \tag{1}$$

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then we say that the tail Π_n decays exponentially and call the θ the exponential decay rate of Π_n. In this paper, we investigate conditions for the exponential decay of Π_n and determine the exponential decay rate θ.

This line of active research has been motivated by the need to evaluate a cell loss probability as small as 10^{-10} in ATM networks. In Ref.[1], based on large deviation theory, for the stationary queue length Q, an upper bound of the form $P(Q > n) \leq \psi \exp(-\theta n)$ is obtained, where ψ is a constant called an asymptotic constant. When a large deviation condition is satisfied, it is shown that $P(Q > n)$ decays as $\lim_{n\to\infty} P(Q > n)\exp(\theta n) = \psi$. In Refs.[6,7], the stationary distribution of M/G/1 or G/M/1 type Markov chains are deeply studied. In Ref.[9], the tail of the waiting time in PH/PH/c queue is investigated. In Ref.[2], sufficient conditions are given for the stationary distribution $\pi = (\pi_n)$ of an M/G/1 type Markov chain with boundary modification to decay as $\pi_n \simeq K r^{-n} + O(\tilde{r}^{-n}),\ K > 0,\ 0 < r < \tilde{r}$. The result is applied to MAP/G/1 queues.

In this paper, we first consider a sequence of complex numbers, or a complex sequence, for short. We define the tail of a complex sequence and give a sufficient condition for the exponential decay of the tail of the complex sequence. We focus on analytic properties of the power series whose coefficients are the given complex sequence such as the radius of convergence and the number of poles on the circle of convergence. The results are of course applicable to discrete probability distributions. We give a counter example for the Proposition 1 in Ref.[3], which claims a stronger result than in this paper. Furthermore, we give an example of an M/G/1 type Markov chain such that the tail of its stationary distribution does not decay exponentially.

2. ASYMPTOTIC BEHAVIOR OF THE TAIL OF DISCRETE PROBABILITY DISTRIBUTION

2.1. Cauchy–Hadamard's Theorem

In general, for a complex sequence $c = (c_n)_{n\geq 0}$, let us consider a power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{2}$$

and study the asymptotic behavior of $c = (c_n)$ according to analytic properties of $f(z)$. The basic tool for this purpose is Cauchy–Hadamard's theorem on the radius of convergence of a power series.

Theorem (Cauchy–Hadamard,[5] p. 344)**.** *Denote by r the radius of convergence of a power series* (2). *Then, we have*

$$\limsup_{n\to\infty} |c_n|^{1/n} = r^{-1}. \tag{3}$$

For a discrete probability distribution $\pi = (\pi_n)$, $\pi_n \geq 0$, $n = 0, 1, \ldots, \sum_{n=0}^{\infty} \pi_n = 1$, the probability generating function $\pi(z)$ of π is defined as a power series;

$$\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n. \tag{4}$$

Since $\pi(1) = 1$, the radius of convergence r of $\pi(z)$ satisfies $r \geq 1$. By Cauchy–Hadamard's theorem, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \pi_n = -\log r. \tag{5}$$

In this paper, we give a condition for the exponential decay of the tail of π. We will show in Theorem 3 that if the radius of convergence r of $\pi(z)$ satisfies $1 < r < \infty$ and $\pi(z)$ has a finite number of poles on the circle of convergence $|z| = r$ and is holomorphic on some neighborhood of $|z| \leq r$ except for the poles, then the tail of π decays exponentially. These conditions hold if $\pi(z)$ is meromorphic on some neighborhood of $|z| \leq r$, so we can say that in almost all realistic cases we have the limit of the form (1).

2.2. Asymptotic Behavior of Complex Sequence

In this section, we investigate the asymptotic behavior of a complex sequence $c = (c_n)$, which is not necessarily a probability vector. The tail C_n of c is defined by $C_n \equiv \sum_{i>n} |c_i|$. If the limit

$$\lim_{n\to\infty} \frac{1}{n} \log C_n = -\theta \tag{6}$$

exists and $0 < \theta < \infty$ holds, then we say that the tail C_n decays exponentially and call the θ the exponential decay rate of C_n. To investigate the asymptotic behavior of C_n, we focus on analytic properties of the power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with coefficients $c = (c_n)$.

In general, if a function $f(z)$ is holomorphic on an open set U except for poles, then $f(z)$ is said to be *meromorphic* on U. For a function $f(z)$, if there exists $\delta > 0$ such that the $f(z)$ is meromorphic on $|z| < r + \delta$, then we say that $f(z)$ is meromorphic on $|z| \leq r$ for the sake of simplicity. Likewise, if there exists $\delta > 0$ such that $f(z)$ is holomorphic on $|z| < r + \delta$, then we say that $f(z)$ is holomorphic on $|z| \leq r$. We have the next lemma.

Lemma 1. *Let r be the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $0 < r < \infty$. Then, $f(z)$ is meromorphic on $|z| \leq r$ if and only if the singularities of $f(z)$ on the circle of convergence $|z| = r$ are only a finite number of poles.*

For a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with the radius of convergence r, from Caucy–Hadamard's theorem, we see that there exists a subsequence $\{n_k\}$ with $\lim_{k\to\infty} |c_{n_k}|^{1/n_k} = r^{-1}$. We investigate the asymptotic behavior of $c = (c_n)$ by considering this subsequence $\{n_k\}$.

Denote by $\mathbb{N}$ the set of natural numbers, i.e., $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^+$ the set of positive natural numbers.

Theorem 1. *Let r denote the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with $0 < r < \infty$. Assume that $f(z)$ is meromorphic on $|z| \leq r$. Let m denote the number of poles of $f(z)$ on $|z| = r$. Then there exist $m^* \in \mathbb{N}^+$, $1 \leq m^* \leq m$, and a sequence $\{n_k\}_{k\geq 0} \subset \mathbb{N}$ which satisfy*

$$\lim_{k\to\infty} |c_{n_k}|^{1/n_k} = r^{-1}, \tag{7}$$

$$km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \dots. \tag{8}$$

Proof. See Appendix.

Remark 1. From the proof of Theorem 1, we see that m^* is the number of poles with the largest order. Hence from Theorem 1, if $m^* = 1$, then $\lim_{n\to\infty} |c_n|^{1/n} = r^{-1}$ holds.

From Theorem 1, we have

Theorem 2. *Let r be the radius of convergence of a power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$, and define $C_n = \sum_{i>n} |c_i|$. If $1 < r < \infty$ and $f(z)$ is meromorphic on $|z| \leq r$, then we have*

$$\lim_{n\to\infty} C_n^{1/n} = r^{-1}. \tag{9}$$

Proof. First we show

$$\limsup_{n\to\infty} C_n^{1/n} \leq r^{-1}. \tag{10}$$

In fact, by Cauchy–Hadamard's theorem, for any ϵ, $0 < \epsilon < 1 - r^{-1}$, there exists $n_0 \in \mathbb{N}$ such that $|c_n|^{1/n} \leq r^{-1} + \epsilon$ holds for any $n \geq n_0$. Thus, we have $C_n \leq (r^{-1} + \epsilon)^{n+1}/(1 - r^{-1} - \epsilon)$ which leads to (10).

Next, we show

$$\liminf_{n\to\infty} C_n^{1/n} \geq r^{-1}. \tag{11}$$

Let m be the number of poles of $f(z)$ on $|z| = r$. Then, by Theorem 1, there exists $m^* \in \mathbb{N}^+$, $1 \leq m^* \leq m$, such that for any ϵ, $0 < \epsilon < r^{-1}$, and any $n \geq n_0$, there is at least one, say, c_{i_0}, among $c_{n+1}, \dots, c_{n+m^*}$ that satisfies $|c_{i_0}|^{1/i_0} \geq r^{-1} - \epsilon$. Therefore, $C_n \geq |c_{i_0}| \geq (r^{-1} - \epsilon)^{i_0} \geq (r^{-1} - \epsilon)^{n+m^*}$, and hence (11) holds.

The next theorem is an immediate consequence of Theorem 2.

Theorem 3. *Let $\pi = (\pi_n)$ be a probability distribution and define the tail of π by $\Pi_n = \sum_{i>n} \pi_i$. Denote by r the radius of convergence of the probability generating function $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$ of π. If $1 < r < \infty$ and $\pi(z)$ is meromorphic on $|z| \leq r$, then we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \Pi_n = -\log r. \tag{12}$$

Remark 2. By Pringsheim's theorem (Theorem 17.13 of Ref.[5]), we see that $z = r$ is a singularity of $\pi(z)$.

2.3. A Counter Example When the Number of Poles is not Finite

In Theorem 3, we assumed that the number of poles of $\pi(z)$ on the circle of convergence $|z| = r$ is finite. We will show below an example of a distribution $\pi = (\pi_n)$ whose tail does not decay exponentially and $\pi(z)$ has infinite number of singularities on the circle of convergence.

Let $\pi = (\pi_n)$ be a probability distribution and put $\Pi_n = \sum_{i>n} \pi_i$. For $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$ and $\Pi(z) = \sum_{n=0}^{\infty} \Pi_n z^n$, we have the following lemma.

Lemma 2. *For $x > 1$, $\pi(x) < \infty$ and $\Pi(x) < \infty$ are equivalent.*

Proof. Since $\pi_0 = 1 - \Pi_0$, $\pi_n = \Pi_{n-1} - \Pi_n$, $n = 1, 2, \ldots$, we have for $N \in \mathbb{N}^+$

$$\sum_{n=0}^{N} \pi_n x^n = 1 + (x-1) \sum_{n=0}^{N-1} \prod_n x^n - \prod_N x^N. \tag{13}$$

Hence, $\Pi(x) < \infty$ implies $\pi(x) < \infty$. Conversely, if $\pi(x) < \infty$, we have $\infty > \sum_{n=N+1}^{\infty} \pi_n x^n \geq x^{N+1} \sum_{n=N+1}^{\infty} \pi_n = x^{N+1} \Pi_N$, which implies $\Pi(x) < \infty$.

From Lemma 2, we have the following lemma.

Lemma 3. *If $\pi(z)$ has the radius of convergence r with $1 < r < \infty$, then the radius of convergence of $\Pi(z)$ is also r, and vice versa.*

Now, for an integer $h \geq 2$, we define a sequence $\{v_k\}_{k\geq 0} \subset \mathbb{N}$ by

$$\begin{cases} v_0 = 0, \\ v_{k+1} = v_k + h^{v_k}, \quad k = 0, 1, \ldots, \end{cases} \tag{14}$$

and for $n \in \mathbb{N}$, we define a function $\phi(n) \in \mathbb{R}$ by

$$\phi(n) = h^{-v_k}, \quad \text{for } v_k \leq n < v_{k+1}, \tag{15}$$

where $\mathbb{R}$ is the set of real numbers. $\phi(n)$ has the following properties;

$$\phi(0) = 1, \quad \phi(n) \geq \phi(n+1), \quad n = 0, 1, \ldots, \tag{16}$$

$$\sum_{n=v_k}^{v_{k+1}-1} \phi(n) = h^{-v_k}(v_{k+1} - v_k) = 1, \quad k = 0, 1, \ldots, \tag{17}$$

$$\sum_{n=0}^{\infty} \phi(n) = \infty, \tag{18}$$

$$\sum_{n=0}^{\infty} \alpha^{-n}\phi(n) < \infty, \quad \text{for any } \alpha > 1. \tag{19}$$

(16)–(18) are easily seen. (19) can be verified as follows. Since $v_k \geq k$ holds by induction, we see from (17),

$$\sum_{n=0}^{\infty} \alpha^{-n}\phi(n) = \sum_{k=0}^{\infty} \sum_{n=v_k}^{v_{k+1}-1} \alpha^{-n}\phi(n) \tag{20}$$

$$\leq \sum_{k=0}^{\infty} \alpha^{-v_k} < \infty. \tag{21}$$

We here define a random variable $T_\alpha \in \mathbb{N}$ for $\alpha > 1$, by a probability distribution function

$$P(T_\alpha \leq n) \equiv 1 - \alpha^{-(n+1)}\phi(n+1), \quad n = 0, 1, \ldots. \tag{22}$$

So, we have

$$P(T_\alpha > n) = \alpha^{-(n+1)}\phi(n+1), \quad n = 0, 1, \ldots. \tag{23}$$

The probability generating function $\tau_\alpha(z)$ of $p_{\alpha,n} \equiv P(T_\alpha = n)$ is defined by

$$\tau_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{\alpha,n} z^n, \tag{24}$$

and the generating function $T_\alpha(z)$ of $P(T_\alpha > n)$ by

$$T_\alpha(z) \equiv \sum_{n=0}^{\infty} P(T_\alpha > n) z^n. \tag{25}$$

Then, by (18), (19), we have $T_\alpha(\alpha) = \infty$ and $T_\alpha(\alpha') < \infty$ for any $\alpha' < \alpha$, so, it follows that α is the radius of convergence of $T_\alpha(z)$. By Cauchy–Hadamard's theorem, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log P(T_\alpha > n) = -\log \alpha. \tag{26}$$

On the other hand, for $n = v_k - 1$, we have

$$P(T_\alpha > v_k - 1) = \alpha^{-v_k}\phi(v_k) \tag{27}$$

$$= \alpha^{-v_k} h^{-v_k}, \quad k = 0, 1, \ldots, \tag{28}$$

hence

$$\liminf_{n\to\infty} \frac{1}{n} \log P(T_\alpha > n) \le \liminf_{k\to\infty} \frac{1}{v_k} \log P(T_\alpha > v_k - 1) \tag{29}$$

$$= -\log\alpha - \log h \tag{30}$$

$$< -\log\alpha. \tag{31}$$

From (26), (31), we see that $\lim_{n\to\infty} n^{-1}\log P(T_\alpha > n)$ does not exist. Moreover, from (30), the difference between lim sup and lim inf can be arbitrarily large by taking h large. By Lemma 3, the radius of convergence of $\tau_\alpha(z)$ is equal to α. The α satisfies $1 < \alpha < \infty$, however, the tail of the distribution $(p_{\alpha,n})$ does not decay exponentially.

By comparing this example and Theorem 3, we see that the $\tau_\alpha(z)$ is not meromorphic on $|z| \le \alpha$. In fact, the following theorem describes about the singularities of $\tau_\alpha(z)$ on $|z| = \alpha$.

Theorem (Hadamard's Gap Theorem,[4] Theorem 11.7.1)**.** *If a power series* $f(z) = \sum_{k=0}^{\infty} c_{v_k} z^{v_k}$ *satisfies*

$$\liminf_{k\to\infty} \frac{v_{k+1}}{v_k} > 1, \tag{32}$$

then all the points on the circle of convergence are singularities of $f(z)$.

This theorem can be applied to our example. Define $G(z) = (\alpha^{-1}z - 1)T_\alpha(z)$. Since the power series expansion of $G(z)$ satisfies (32), all the points on the circle of convergence $|z| = \alpha$ are singularities of $G(z)$. From the proof of Lemma 2, $\tau_\alpha(z) = 1 + (z-1)T_\alpha(z)$, and hence, we have $\tau_\alpha(z) = 1 + (z-1)G(z)/(\alpha^{-1}z - 1)$, which implies that all the points on $|z| = \alpha$ are singularities of $\tau_\alpha(z)$.

Remark 3. Equation (23) is a counter example for Proposition 1 in Ref.[3], which overlooks the need for the extra assumption, say, the number of poles on the circle of convergence.

3. EXAMPLE OF M/G/1 TYPE MARKOV CHAIN SUCH THAT THE TAIL OF ITS STATIONARY DISTRIBUTION DOES NOT DECAY EXPONENTIALLY

In this section, we show an example of an M/G/1 type Markov chain such that the tail of its stationary distribution does not decay exponentially.

Consider a Markov chain with the following transition matrix P.

$$P = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{33}$$

For the probability vectors $A \equiv (a_0, a_1, \dots)$, $B \equiv (b_0, b_1, \dots)$, define $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$. If $E(A) \equiv \sum_{n=0}^{\infty} n a_n < 1$ and $E(B) \equiv \sum_{n=0}^{\infty} n b_n < \infty$, then there exists the stationary distribution $\pi = (\pi_n)$ of P. Define $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$, then we have

$$\pi(z)(z - A(z)) = \pi_0(zB(z) - A(z)). \tag{34}$$

Now, let $A = (1-q, q, 0, 0, \dots)$ with $0 < q < 1$, which represents a Bernoulli process. We have $A(z) = 1 - q + qz$. Then define

$$B(z) \equiv 1 + z^{-1}(p_{\alpha,0}^{-1}\tau_\alpha(z) - 1)(z - A(z)) \tag{35}$$

$$= 1 + (1-q)z^{-1}(p_{\alpha,0}^{-1}\tau_\alpha(z) - 1)(z - 1), \tag{36}$$

where $p_{\alpha,0}$ and $\tau_\alpha(z)$ were defined in 2.3. Substituting (36) into (34), we have $\pi(z) = \tau_\alpha(z)$. We should verify that the coefficients $B = (b_0, b_1, \dots)$ of $B(z)$ satisfy $b_n \geq 0$, $n = 0, 1, \dots$. In fact, we can show that $b_n > 0$, $n = 0, 1, \dots$ for $\alpha \geq 2$.

To prove it, we need the next lemma.

Lemma 4. *For $\alpha \geq 2$, $p_{\alpha,n}$ decreases monotonically, i.e., $p_{\alpha,n} > p_{\alpha,n+1}$, $n = 0, 1, \dots$.*

Proof. Since $p_{\alpha,n} = P(T_\alpha > n-1) - P(T_\alpha > n) = \alpha^{-n}\phi(n) - \alpha^{-(n+1)}\phi(n+1)$, $n = 0, 1, \dots$, we have $p_{\alpha,n} - p_{\alpha,n+1} = \alpha^{-(n+2)}(\alpha^2\phi(n) - 2\alpha\phi(n+1) + \phi(n+2))$. Thus, from $\alpha^2 \geq 2\alpha$, $\phi(n) > 0$, and $\phi(n) \geq \phi(n+1)$, we have $p_{\alpha,n} > p_{\alpha,n+1}$.

By comparing the coefficients of expansion of the both sides of (36), we have, by Lemma 4, for $\alpha \geq 2$,

$$b_0 = 1 - (1-q)p_{\alpha,0}^{-1}p_{\alpha,1} > 0, \tag{37}$$

$$b_n = (1-q)p_{\alpha,0}^{-1}(p_{\alpha,n} - p_{\alpha,n+1}) > 0, \quad n = 1, 2, \dots. \tag{38}$$

In summary, we have

Theorem 4. *For $A = (1-q, q, 0, 0, \dots)$, $0 < q < 1$, and $B = (b_0, b_1, \dots)$ defined by* (36) *with $\alpha \geq 2$, the matrix P of* (33) *is irreducible and positive recurrent. The tail of the stationary distribution of P does not decay exponentially.*

Remark 4. The example in Theorem 4 does not satisfy the condition (i) of Theorem 3.5 in Ref.[2]. Is it possible to make an example of an M/G/1 type Markov chain which satisfies the conditions (i), (ii) of Theorem 3.5 in Ref.[2] but the tail of its stationary distribution does not decay exponentially?

4. CONCLUSION

In this paper, based on analytic properties of the probability generating function $\pi(z)$ of a discrete probability distribution π, we proved that if the radius of convergence r of $\pi(z)$ satisfies $1 < r < \infty$ and $\pi(z)$ is meromorphic on $|z| \leq r$, then the tail of π decays exponentially. We displayed a counter example for Proposition 1 in Ref.[3]. We also gave an example of an $M/G/1$ type Markov chain P such that the tail of the stationary distribution of P does not decay exponentially.

The examples shown in this paper of probability generating functions whose tail probabilities do not decay exponentially are pathological in the sense that all the points on the circle of convergence are singularities. In almost all the realistic cases, $\pi(z)$ is expected to be a meromorphic function, and therefore, we can obtain the exponential decay rate of the tail only by calculating the radius of convergence.

5. APPENDIX

Proof of Theorem 1

First, we need the following lemma. Denote by $\mathbb{C}$ the set of complex numbers, $\mathbb{N}$ the set of natural numbers, and $\mathbb{N}^+$ the set of positive natural numbers.

Lemma 5. *For $m \in \mathbb{N}^+$, consider complex numbers $\beta_i,\ w_i \in \mathbb{C},\ i = 1, \ldots, m$, with the following properties*;

$$\beta_i \neq 0, \quad |w_i| = 1, \ w_i \neq w_j \ \text{ for } i \neq j, \ i, j = 1, \ldots, m. \tag{39}$$

Define $t_n \equiv \sum_{i=1}^{m} \beta_i w_i^n$ for $n \in \mathbb{N}$. Then there exists an $\epsilon > 0$ such that for any $N \in \mathbb{N}$ there is at least one n among consecutive m natural numbers $N, N+1, \ldots, N+m-1$ with $|t_n| > \epsilon$.

Proof. Suppose that the claim were not true. Then for any $\epsilon > 0$ there must exist an $N \in \mathbb{N}$ such that $|t_n| \leq \epsilon$ holds for $n = N, N+1, \ldots, N+m-1$. Consider the matrix representation

$$\begin{pmatrix} t_N \\ t_{N+1} \\ \vdots \\ t_{N+m-1} \end{pmatrix} = \begin{pmatrix} w_1^N & w_2^N & \ldots\ w_m^N \\ w_1^{N+1} & w_2^{N+1} & \ldots\ w_m^{N+1} \\ & \ldots & \\ w_1^{N+m-1} & w_2^{N+m-1} & \ldots\ w_m^{N+m-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}. \tag{40}$$

Let V_N be the matrix of the right hand side of (40). We have

$$\begin{aligned} \det V_N &= w_1^N w_2^N \cdots w_m^N \times \det \begin{pmatrix} 1 & 1 & \ldots\ 1 \\ w_1 & w_2 & \ldots\ w_m \\ & \ldots & \\ w_1^{m-1} & w_2^{m-1} & \ldots\ w_m^{m-1} \end{pmatrix} \\ &= w_1^N w_2^N \cdots w_m^N (-1)^{m(m-1)/2} \prod_{i<j} (w_i - w_j), \end{aligned}$$

(Vandermonde's determinant) and hence

$$|\det V_N| = \prod_{i<j} |w_i - w_j| \equiv D > 0. \tag{41}$$

Denoting by Δ_{ij} the adjoint of V_N, we have

$$|\Delta_{ij}| \le (m-1)!, \quad i, j = 1, \ldots, m, \tag{42}$$

because Δ_{ij} is the sum of $(m-1)!$ terms of absolute value 1. From (40), we have

$$\beta_i = \frac{1}{\det V_N} \sum_{j=1}^{m} \Delta_{ij} t_{N+j-1}, \quad i = 1, \ldots, m, \tag{43}$$

thus,

$$|\beta_i| \le \frac{1}{D} \sum_{j=1}^{m} |\Delta_{ij}||t_{N+j-1}| \tag{44}$$

$$\le \frac{m!}{D}\epsilon, \quad i = 1, \ldots, m. \tag{45}$$

Since $\epsilon > 0$ is arbitrary, from (45), we have $\beta_i = 0$, but this contradicts the assumption of the lemma.

Next, we prove the following proposition which corresponds to the case $r = 1$ of Theorem 1.

Proposition 1. *Assume that the radius of convergence of* $f(z) = \sum_{n=0}^{\infty} c_n z^n$ *is equal to* 1, *and* $f(z)$ *is meromorphic on* $|z| \le 1$. *Let* m *be the number of poles on* $|z| = 1$. *Then, there exist* $m^* \in \mathbb{N}^+$, $1 \le m^* \le m$, *and a sequence* $\{n_k\}_{k \ge 0} \subset \mathbb{N}$ *which satisfy*

$$\lim_{k\to\infty} |c_{n_k}|^{1/n_k} = 1, \tag{46}$$

$$km^* \le n_k < (k+1)m^*, \quad k = 0, 1, \ldots. \tag{47}$$

Proof (cf.[8], p.152, problem 242). Let $\zeta_1, \ldots, \zeta_m$ be the poles of $f(z)$ on the circle of convergence $|z| = 1$, and s_i the order of the pole ζ_i, $i = 1, \ldots, m$. Let $p_i(z)$ be the principal part of $f(z)$ at ζ_i with

$$p_i(z) = \frac{1}{(\zeta_i - z)^{s_i}} \sum_{l=0}^{s_i - 1} a_{il} z^l, \quad i = 1, \ldots, m. \tag{48}$$

Since $\sum_{l=0}^{s_i-1} a_{il} z^l$ does not have a factor $\zeta_i - z$, we see $\sum_{l=0}^{s_i-1} a_{il}\zeta_i^l \ne 0$, $i = 1, \ldots, m$. Put $g(z) \equiv f(z) - \sum_{i=1}^{m} p_i(z)$, then $g(z)$ is holomorphic on $|z| \le 1$. Let $g(z) = \sum_{n=0}^{\infty} d_n z^n$ be the power series expansion of $g(z)$, then by Cauchy–Hadamard's theorem, we have $\limsup_{n\to\infty} |d_n|^{1/n} < 1$. Let $p_i(z) = \sum_{n=0}^{\infty} e_{in} z^n$ be the expansion of $p_i(z)$. Using the expansion

$$\frac{1}{(\zeta - z)^s} = \frac{1}{\zeta^s} \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} \zeta^{-n} z^n, \tag{49}$$

we have

$$e_{in} = \sum_{l=0}^{s_i-1} \binom{n-l+s_i-1}{s_i-1} \zeta_i^{-s_i-n+l} a_{il}, \tag{50}$$

for $n \geq s_i - 1$, $i = 1, \ldots, m$. Then by $f(z) = \sum_{i=1}^{m} p_i(z) + g(z)$, we have

$$c_n = \sum_{i=1}^{m} \sum_{l=0}^{s_i-1} \binom{n-l+s_i-1}{s_i-1} \zeta_i^{-s_i-n+l} a_{il} + d_n, \tag{51}$$

for $n \geq \max_{1\leq i\leq m}(s_i - 1)$.

Since $\binom{n-l+s_i-1}{s_i-1} = n^{s_i-1}(1 + O(n^{-1}))$, we have, from (50),

$$e_{in} = n^{s_i-1}\left(\zeta_i^{-s_i-n} \sum_{l=0}^{s_i-1} \zeta_i^l a_{il}\right)(1 + O(n^{-1})). \tag{52}$$

Let $S \equiv \max_{1\leq i\leq m} s_i$. Without loss of generality, we can assume that $i = 1, \ldots, m^*(\leq m)$ attain the maximum. From (51), we have

$$c_n = n^{S-1}\left(\sum_{i=1}^{m^*} \zeta_i^{-S-n} \sum_{l=0}^{S-1} a_{il}\zeta_i^l\right)(1 + O(n^{-1})), \tag{53}$$

and then

$$|c_n|^{1/n} = (n^{1/n})^{S-1} \left|\sum_{i=1}^{m^*} \zeta_i^{-S-n} \sum_{l=0}^{S-1} a_{il}\zeta_i^l\right|^{1/n} \left|1 + O(n^{-1})\right|^{1/n}. \tag{54}$$

In the right hand side of (54), we see that $\lim_{n\to\infty}(n^{1/n})^{S-1}=1$, $\lim_{n\to\infty}\left|1 + O(n^{-1})\right|^{1/n} = 1$. Meanwhile, about the second factor of (54), writing

$$w_i \equiv \zeta_i^{-1}, \quad \beta_i \equiv \zeta_i^{-S} \sum_{l=0}^{S-1} a_{il}\zeta_i^l, \ \ i = 1, \ldots, m^*, \tag{55}$$

we have

$$\beta_i \neq 0, \quad |w_i| = 1, \quad w_i \neq w_j \quad \text{for } i \neq j, \quad i, j = 1, \ldots, m^*. \tag{56}$$

Then by Lemma 5, there exist $\epsilon > 0$ and a sequence $\{n_k\} \subset \mathbb{N}$, such that

$$\left|\sum_{i=1}^{m^*} \zeta_i^{-S-n_k} \sum_{l=0}^{S-1} a_{il}\zeta_i^l\right| > \epsilon, \tag{57}$$

$$km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \ldots.$$

In summary, we have

$$\lim_{k\to\infty} \left|c_{n_k}\right|^{1/n_k} = 1, \tag{58}$$

$$km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \ldots. \tag{59}$$

Proof of Theorem 1. Apply Proposition 1 to the power series $\sum_{n=0}^{\infty} c_n r^n z^n$ with the radius of convergence 1.

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