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TAIL PROBABILITY AND SINGULARITY OF LAPLACE-STIELTJES TRANSFORM OF A PARETO TYPE RANDOM VARIABLE

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Abstract. We give a sufficient condition for a non-negative random variable X to be of Pareto type by investigating the Laplace-Stieltjes transform of the cumulative distribution function. We focus on the relation between the singularity at the real point of the axis of convergence and the asymptotic decay of the tail probability. For the proof of our theorems, we apply Graham-Vaaler's complex Tauberian theorem. As an application of our theorems, we consider the asymptotic decay of the stationary distribution of an M/G/1 type Markov chain.

Keywords: tail probability; Pareto type; Laplace-Stieltjes transform; Tauberian theorem

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1. Introduction

We consider the asymptotic decay of the *tail probability* $P(X > x)$ of a Pareto type random variable X. A random variable X is said to be of *Pareto type with decay exponent* $r > 0$ if

$$\frac{C'}{x^r} \leqslant P(X > x) \leqslant \frac{C}{x^r} \tag{1.1}$$

holds for all sufficiently large x, where C and C' are positive constants.

Denote by $F(x)$ the cumulative distribution function of X, i.e., $F(x) = P(X \leqslant x)$. For example, let X follow a Pareto distribution such that $F(x) = 1 - 1/x$, $x \geqslant 1$, then $P(X > x) = 1/x$, so X is of Pareto type with decay exponent $r = 1$. Pareto distribution is naturally of Pareto type.

In this paper, we will give a sufficient condition for X to be of Pareto type based on analytic properties of the Laplace-Stieltjes (LS) transform of $F(x)$. The LS transform of $F(x)$ is defined by

$$\varphi(s) = \int_0^\infty e^{-sx}\,dF(x). \tag{1.2}$$

In general, for a function $R(x)$, which is of bounded variation in the interval $0 \leqslant x \leqslant c$ for any positive c, the LS transform $\Psi(s) = \int_0^\infty e^{-sx}\,dR(x)$, $s = \sigma + i\tau$, is defined. If $\Psi(s)$ converges for $\sigma > \sigma_0$ and diverges for $\sigma < \sigma_0$, then σ_0 is said to be the *abscissa of convergence* of $\Psi(s)$. The line $\Re s = \sigma_0$ is called the *axis of convergence*, where $\Re s$ denotes the real part of s. In the case $\sigma_0 = 0$, some local information of $\Psi(s)$ at $s = 0$ may provide the asymptotic behavior of $R(x)$ as $x \to \infty$. Such a proposition is called a Tauberian theorem. The following is one of the Tauberian theorems.

Theorem (Widder [13], p. 192, Theorem 4.3)**.** *Let $R(x)$ be a non-decreasing function and let the abscissa of convergence of $\Psi(s)$ be $\sigma_0 = 0$. If for constants $r \leqslant 0$ and A*

$$\lim_{s\to 0+} |\Psi(s)s^{-r} - A| = 0, \tag{1.3}$$

then

$$\lim_{x\to\infty} |R(x)\Gamma(r+1)x^r - A| = 0, \tag{1.4}$$

where Γ denotes the gamma function.

In [9], [10], [11] we studied the asymptotic decay of a light tailed random variable. A random variable X is said to be *light tailed* if the tail probability $P(X > x)$ decays exponentially, i.e.,

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) < 0. \tag{1.5}$$

We obtained in [11] the following theorem which gives a sufficient condition for a light tailed random variable.

Theorem (Nakagawa [11])**.** *For a non-negative random variable X with cumulative distribution function $F(x)$, let $\varphi(s) = \int_0^\infty e^{-sx}\,dF(x)$ be the Laplace-Stieltjes transform of $F(x)$ and σ_0 the abscissa of convergence of $\varphi(s)$. We assume $-\infty < \sigma_0 < 0$. If $s = \sigma_0$ is a pole of $\varphi(s)$, then we have*

$$\lim_{x\to\infty} \frac{1}{x} \log P(X > x) = \sigma_0. \tag{1.6}$$

It is known that for a monotonic $R(x)$, $s = \sigma_0$ is a singularity of $\Psi(s)$ (see Widder [13], p. 58, Theorem 5b). Since the cumulative distribution function $F(x)$ is monotonic increasing, $s = \sigma_0$ is a singularity of $\varphi(s)$. If $F(x)$ is the cumulative distribution function of a Pareto type random variable, then the abscissa of convergence of $\varphi(s)$ is necessarily $\sigma_0 = 0$ (see Widder [13], p. 40, Theorem 2.2b). Since $\varphi(0) = \int_0^\infty \mathrm{d}F(x) = 1$, $s = 0$ is not a pole, but another type of singularity.

In the research of the asymptotic decay of a tail probability, we would like to construct a general theory such that a local analytic information on $\varphi(s)$ at $s = \sigma_0$ tells the asymptotic evaluation of the tail probability. In a light tailed case [9], [10], [11] we applied to this problem Ikehara's Tauberian theorem [6], [7] and its extension Graham-Vaaler's Tauberian theorem [5], [7]. Ikehara's theorem assumes a global analytic property of $\varphi(s)$, that is, $s = \sigma_0$ is a pole and there exist no other singularities on the axis of convergence $\Re s = \sigma_0$. On the other hand, Graham-Vaarler's theorem only assumes $s = \sigma_0$ is a pole, which yields a weaker assertion than Ikehara's theorem, however, it is enough for our purpose to investigate the asymptotic decay of a tail probability. In this paper, we will apply Graham-Vaaler's Tauberian theorem to the decay of a Pareto type random variable.

We will consider an application of our theorems to an M/G/1 type Markov chain [3] to obtain a simple criterion for the stationary distribution to be of Pareto type; see Chapter 7. In an M/G/1 type Markov chain, the probability generating function $\pi(z)$ of the stationary distribution $\pi = (\pi_n)$ is algebraically represented by the probability generating functions of the rows of the state transition probability matrix [3]. Therefore, we can obtain the stationary probabilities by the inverse z-transform of $\pi(z)$, however, the inverse transform is difficult in general. Because our purpose is to know the asymptotic decay of the probabilities, we do not need to know the exact value of π_n. If we know the singularity of $\pi(z)$, then we can apply our theorems to obtain the asymptotic behavior of π_n.

Throughout this paper, we will use the following symbols. $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$, $\mathbb{C}$, denote the set of natural numbers, positive natural numbers, real numbers, complex numbers, and further, $\Re$, $\mathcal{L}$, $\mathcal{F}$ denote the real part of a complex number, Laplace transform and Fourier transform, respectively.

2. Examples of Pareto type random variables

Now, we look at some examples of Pareto type random variables and their LS transforms.

2.1. Continuous random variable I. Let X follow a Pareto distribution with cumulative distribution function $F(x) = 1 - 1/x$, $x \geqslant 1$. Then X is of Pareto type with decay exponent $r = 1$. The LS transform $\varphi(s)$ of $F(x)$ is represented in a neighborhood of $s = 0$ as

$$\varphi(s) = \int_1^\infty \mathrm{e}^{-sx}\,\mathrm{d}F(x) = s\log s + \beta(s), \tag{2.1}$$

where $\beta(s)$ is analytic in a neighborhood of $s = 0$ (see Lemma 4.1).

2.2. Continuous random variable II. Let X follow a Pareto distribution with cumulative distribution function $F(x) = 1 - 1/\sqrt{x}$, $x \geqslant 1$. Then X is of Pareto type with decay exponent $r = 1/2$ and

$$\varphi(s) = \frac{2\pi}{\Gamma(1/2)}\sqrt{s} + \beta(s), \tag{2.2}$$

where $\beta(s)$ is analytic in a neighborhood of $s = 0$ (see Lemma 4.2).

2.3. Discrete random variable. Let X be a discrete random variable with the probability distribution $p = (p_n)_{n\in\mathbb{N}}$,

$$p_n = \frac{1}{(n+1)^r} - \frac{1}{(n+2)^r}, \quad n \in \mathbb{N},\ r \in \mathbb{N}^+. \tag{2.3}$$

The tail probability of X is $P(X > n) = 1/(n+2)^r$, so X is of Pareto type with decay exponent r. The probability generating function $p(z)$ of p is

$$p(z) = \frac{z-1}{z^2}\sum_{n=1}^{\infty}\frac{z^n}{n^r} + \frac{1}{z}. \tag{2.4}$$

Substituting $z = \mathrm{e}^{-s}$, we have

$$\varphi(s) = p(\mathrm{e}^{-s}) = \alpha(s)s^r\log s + \beta(s), \tag{2.5}$$

where $\alpha(s)$ and $\beta(s)$ are analytic in a neighborhood of $s = 0$ with $\alpha(0) \neq 0$ (see Lemma 4.3).

3. Main theorems

The following are the main theorems in this paper.

Theorem 3.1. *Let X be a non-negative random variable with cumulative distribution function $F(x)$, and let $\varphi(s)$ be the Laplace-Stieltjes transform of $F(x)$. Assume the abscissa of convergence of $\varphi(s)$ is $\sigma_0 = 0$, and $\varphi(s)$ is represented in a neighborhood of $s = 0$ as*

$$\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+, \tag{3.1}$$

where $\alpha(s)$ and $\beta(s)$ are analytic functions with $\alpha(0) \neq 0$. Then X is of Pareto type with decay exponent r, i.e., there exist constants $C > 0$, $C' > 0$ such that

$$\frac{C'}{x^r} \leqslant P(X > x) \leqslant \frac{C}{x^r} \tag{3.2}$$

for all sufficiently large x.

Theorem 3.2. *Under the same notation as in Theorem 3.1, assume $\sigma_0 = 0$ and $\varphi(s)$ is represented in a neighborhood of $s = 0$ as*

$$\varphi(s) = \alpha(s)s^r + \beta(s), \quad r > 0,\ r \notin \mathbb{N}, \tag{3.3}$$

where $\alpha(s)$ and $\beta(s)$ are analytic functions with $\alpha(0) \neq 0$. Then X is of Pareto type with decay exponent r, i.e., there exist constants $C > 0$, $C' > 0$ such that

$$\frac{C'}{x^r} \leqslant P(X > x) \leqslant \frac{C}{x^r} \tag{3.4}$$

for all sufficiently large x.

The following result is related to our theorems. Under the same notation as in Theorem 3.1, assume $\sigma_0 = 0$, and denote by μ_n the nth moment of X, i.e., $\mu_n = E(X^n)$, $n \in \mathbb{N}$. For $n \in \mathbb{N}$ with $\mu_n < \infty$, define

$$\varphi_n(s) = (-1)^{n+1}\left\{\varphi(s) - \sum_{k=0}^{n} \frac{(-1)^k \mu_k}{k!} s^k\right\}, \quad s > 0, \tag{3.5}$$

and

$$\tilde{\varphi}_n(s) = \frac{\mathrm{d}^n \varphi_n(s)}{\mathrm{d}s^n} = \mu_n - (-1)^n \varphi^{(n)}(s), \quad s > 0, \tag{3.6}$$

where $\varphi^{(n)}(s)$ denotes the n-th derivative of $\varphi(s)$.

For functions $f_1(x)$ and $f_2(x)$, we write $f_1(x) \sim f_2(x)$ as $x \to \infty$ (or $x \to 0$) when $\lim_{x\to\infty \text{ (or } x\to 0)} f_1(x)/f_2(x) = 1$.

Theorem (Bingham, et al. [1], p. 716, Theorem A, see also [2], Theorem 8.1.6)**.** *For $n \in \mathbb{N}$ with $\mu_n < \infty$, let $r = n + \varepsilon$, $0 \leqslant \varepsilon \leqslant 1$, and let $T(x)$ be a slowly varying function (see [2]) as $x \to \infty$. Then the following conditions are equivalent:*

$$\text{(A)} \qquad \varphi_n(s) \sim s^r T(s^{-1}), \quad s \to 0,$$

$$\text{(B)} \qquad \tilde{\varphi}_n(s) \sim \frac{\Gamma(r+1)}{\Gamma(\varepsilon+1)} s^{\varepsilon} T(s^{-1}), \ s \to 0,$$

$$\text{(C1)} \qquad \int_x^{\infty} t^n \, \mathrm{d}F(t) \sim n!\, T(x), \quad x \to \infty, \ \textit{if } \varepsilon = 0,$$

$$\text{(C2)} \qquad 1 - F(x) \sim \frac{(-1)^n}{\Gamma(1-r)} x^{-r} T(x), \quad x \to \infty, \ \textit{if } 0 < \varepsilon < 1,$$

$$\text{(C3)} \qquad \int_0^{x} t^{n+1} \, \mathrm{d}F(t) \sim (n+1)!\, T(x), \quad x \to \infty, \ \textit{if } \varepsilon = 1.$$

If $\varepsilon > 0$, they are also equivalent to

$$(-1)^{n+1} \varphi^{(n+1)}(s) \sim \frac{\Gamma(r+1)}{\Gamma(\varepsilon)} s^{\varepsilon-1} T(s^{-1}), \quad s \to 0.$$

This theorem is based on the theory of regularly varying functions, especially Karamata's theory, see [2], [4], [7]. The theorem asserts the equivalence between the limit of LS transform as $s \to 0$ and that of the cumulative distribution function as $x \to \infty$, whereas our theorems give upper and lower estimates of $P(X > x)$ for all sufficiently large x. Particularly, in (C2), the authors consider the tail probability $P(X > x) = 1 - F(x)$, but the estimation for $P(X > x)$ in our theorems is stronger than that in Bingham's paper.

4. Preliminary lemmas

We will prepare some lemmas for the proof of our main theorems. First, let us define the step function $\Delta(t)$ as

$$\Delta(t) = \begin{cases} 1, & \text{if } t \geqslant 1, \\ 0, & \text{if } t < 1. \end{cases} \tag{4.1}$$

Lemma 4.1. *For $r \in \mathbb{N}^+$, we have*

$$\varphi(s) \equiv \mathcal{L}\Big(\frac{1}{t^{r+1}} \Delta(t)\Big) = \int_1^{\infty} \frac{1}{t^{r+1}} \mathrm{e}^{-st} \, \mathrm{d}t = \frac{(-1)^{r+1}}{r!} s^r \log s + \beta(s), \tag{4.2}$$

where $\beta(s)$ is analytic in a neighborhood of $s = 0$.

P r o o f. The $(r+1)$-st derivative of $\varphi(s)$ is

$$\varphi^{(r+1)}(s) = (-1)^{r+1} \int_1^\infty \mathrm{e}^{-st}\,\mathrm{d}t = (-1)^{r+1}\Big(\frac{1}{s} + \frac{\mathrm{e}^{-s}-1}{s}\Big). \tag{4.3}$$

Since $(\mathrm{e}^{-s}-1)/s$ is analytic, we have, by successive integrations, the desired result. $\square$

Lemma 4.2. *For $r > 0$, $r \notin \mathbb{N}$, let $r_0 = \lfloor r \rfloor$ be the maximum integer among integers smaller than r, and let $\bar{r} = r - r_0$. Then we have*

$$\varphi(s) \equiv \mathcal{L}\Big(\frac{1}{t^{r+1}}\Delta(t)\Big) = \frac{(-1)^{r_0+1}\pi}{\Gamma(r+1)\sin \pi\bar{r}} s^r + \beta(s), \tag{4.4}$$

where $\beta(s)$ is analytic in a neighborhood of $s = 0$.

P r o o f. By a formula of Laplace transform (see [8], p. 287),

$$\begin{aligned} \varphi^{(r_0+1)}(s) &= (-1)^{r_0+1} \int_1^\infty t^{-\bar{r}}\mathrm{e}^{-st}\,\mathrm{d}t \\ &= (-1)^{r_0+1}\Big\{\int_0^\infty - \int_0^1\Big\} t^{-\bar{r}}\mathrm{e}^{-st}\,\mathrm{d}t \\ &= (-1)^{r_0+1}\Gamma(1-\bar{r})s^{\bar{r}-1} + \tilde{\beta}(s), \end{aligned} \tag{4.5}$$

where $\tilde{\beta}(s)$ is analytic in a neighborhood of $s = 0$. By successive integrations,

$$\begin{aligned} \varphi(s) &= (-1)^{r_0+1}\Gamma(1-\bar{r})\frac{1}{\bar{r}} \cdot \frac{1}{\bar{r}+1} \cdot \ldots \cdot \frac{1}{\bar{r}+r_0} s^{\bar{r}+r_0} + \beta(s) \\ &= \frac{(-1)^{r_0+1}\pi}{\Gamma(r+1)\sin \pi\bar{r}} s^r + \beta(s). \end{aligned} \tag{4.6}$$

In (4.6), we applied the formulas $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$. $\square$

Lemma 4.3. *Let X be a discrete random variable with probability distribution $p = (p_n)_{n\in\mathbb{N}}$,*

$$p_n = \frac{1}{(n+1)^r} - \frac{1}{(n+2)^r}, \quad n \in \mathbb{N},\ r \in \mathbb{N}^+, \tag{4.7}$$

and let $p(z) = \sum\limits_{n=0}^{\infty} p_n z^n$ be the probability generating function of p. Then $\varphi(s) = p(\mathrm{e}^{-s})$ is represented in a neighborhood of $s = 0$ as

$$\varphi(s) = \alpha(s)s^r \log s + \beta(s), \tag{4.8}$$

where $\alpha(s)$ and $\beta(s)$ are analytic with $\alpha(0) \neq 0$.

P r o o f. For $r \geqslant 2$, by calculation and Riemann's formula (see Widder [13], p. 232),

$$p(z) = \frac{z-1}{z^2}\sum_{n=1}^{\infty}\frac{z^n}{n^r} + \frac{1}{z} = \frac{1}{(r-1)!}\,\frac{z-1}{z^2}\int_0^{\infty}\frac{zt^{r-1}}{\mathrm{e}^t - z}\,\mathrm{d}t + \frac{1}{z}. \tag{4.9}$$

Then,

$$\varphi(s) = \frac{\mathrm{e}^s(1-\mathrm{e}^s)}{(r-1)!}\int_0^{\infty}\frac{t^{r-1}}{\mathrm{e}^{t+s}-1}\,\mathrm{d}t + \mathrm{e}^s. \tag{4.10}$$

By the change of variables $u = \mathrm{e}^{t+s} - 1$, we have for sufficiently small $s > 0$

$$\begin{aligned}
\varphi(s) &= \frac{\mathrm{e}^s(1-\mathrm{e}^s)}{(r-1)!}\int_{\mathrm{e}^s-1}^{\infty}\frac{\{\log(1+u)-s\}^{r-1}}{u(u+1)}\,\mathrm{d}u + \text{analytic function}\\
&= \frac{\mathrm{e}^s(1-\mathrm{e}^s)}{(r-1)!}\int_{\mathrm{e}^s-1}\Big(\frac{1}{u} - \frac{1}{u+1}\Big)\{(-s)^{r-1} + (r-1)(-s)^{r-2}u + \ldots\}\,\mathrm{d}u\\
&\quad + \text{analytic function}\\
&= \frac{\mathrm{e}^s(1-\mathrm{e}^s)}{(r-1)!}(-s)^{r-1}\int_{\mathrm{e}^s-1}\frac{1}{u}\,\mathrm{d}u + \text{analytic function}\\
&= \alpha(s)s^r\log s + \beta(s),
\end{aligned}$$

where

$$\alpha(s) = \frac{(-1)^{r+1}}{(r-1)!}\mathrm{e}^s\frac{\mathrm{e}^s-1}{s}, \quad \alpha(0) = \frac{(-1)^{r+1}}{(r-1)!} \neq 0 \tag{4.11}$$

and $\beta(s)$ is an analytic function.

For $r = 1$, we have

$$p(z) = \frac{1-z}{z^2}\log(1-z) + \frac{1}{z}. \tag{4.12}$$

A similar argument leads to the desired result. □

Lemma 4.4. *For a cumulative distribution function $F(x)$, let*

$$\varphi(s) = \int_0^{\infty}\mathrm{e}^{-sx}\,\mathrm{d}F(x) \tag{4.13}$$

have the abscissa of convergence $\sigma_0 = 0$. If, in a neighborhood of $s = 0$,

$$\varphi(s) = \alpha(s)s^r\log s + \beta(s), \quad r \in \mathbb{N}^+, \tag{4.14}$$

where $\alpha(s)$, $\beta(s)$ are analytic with $\alpha(0) \neq 0$, then,

$$\alpha(0) \begin{cases} > 0, & \text{if } r \text{ is odd,} \\ < 0, & \text{if } r \text{ is even.} \end{cases} \tag{4.15}$$

P r o o f. From (4.13),

$$\varphi^{(r)}(0+) \begin{cases} \leqslant 0, & \text{if } r \text{ is odd,} \\ \geqslant 0, & \text{if } r \text{ is even,} \end{cases} \tag{4.16}$$

while from (4.14) by calculation

$$\varphi^{(r)}(s) = r!\alpha(s)\log s + \theta(s), \tag{4.17}$$

where $\theta(s)$ is a function of s with $|\theta(0+)| < \infty$. Thus, by (4.17),

$$\varphi^{(r)}(0+) = r!\alpha(0) \times (-\infty) + \theta(0+). \tag{4.18}$$

Comparing (4.16), (4.18), we have the conclusion. □

Lemma 4.5. *Under the same notation as in Lemmas 4.2, 4.4, if $\varphi(s)$ has the abscissa of convergence $\sigma_0 = 0$ and*

$$\varphi(s) = \alpha(s)s^r + \beta(s), \quad r > 0,\ r \notin \mathbb{N}, \tag{4.19}$$

with $\alpha(0) \neq 0$, then

$$\alpha(0) \begin{cases} > 0, & \text{if } r_0 = \lfloor r \rfloor \text{ is odd,} \\ < 0, & \text{if } r_0 \text{ is even.} \end{cases} \tag{4.20}$$

P r o o f. From (4.13)

$$\varphi^{(r_0+1)}(0+) \begin{cases} \geqslant 0, & \text{if } r_0 \text{ is odd,} \\ \leqslant 0, & \text{if } r_0 \text{ is even,} \end{cases} \tag{4.21}$$

while from (4.19)

$$\varphi^{(r_0+1)}(s) = \sum_{k=0}^{r_0+1} \binom{r_0+1}{k} \alpha^{(r_0+1-k)}(s) \cdot \frac{\mathrm{d}}{\mathrm{d}s^k} s^r + \beta^{(r_0+1)}(s). \tag{4.22}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s^k}s^r\Big|_{s=0+} = \begin{cases} 0 & \text{for } k = 0, 1, \dots, r_0, \\ \infty & \text{for } k = r_0 + 1, \end{cases} \tag{4.23}$$

we have from (4.22)

$$\varphi^{(r_0+1)}(0+) = \alpha(0) \times (+\infty) + \beta^{(r_0+1)}(0). \tag{4.24}$$

Comparing (4.21), (4.24), we have the result. □

4.1. First several terms of $\alpha(s)$ and $\beta(s)$. We will need later, in the proof of the main theorems, to make Laplace transforms which have the same first several terms as those in the Taylor expansion of $\alpha(s)$ and $\beta(s)$, respectively.

First, consider the case of Theorem 3.1, i.e.,

$$\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+, \tag{4.25}$$

with expansions $\alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n$, $\alpha(0) \neq 0$, and $\beta(s) = \sum_{n=0}^{\infty} \beta_n s^n$.

We will form functions $g^*(t)$ and $h^*(t)$ such that their Laplace transforms $G^*(s) \equiv \mathcal{L}(g^*(t))$ and $H^*(s) \equiv \mathcal{L}(h^*(t))$ satisfy the following (i) and (ii) for $L \in \mathbb{N}^+$:

(i) $G^*(s) = \alpha^*(s)s^r \log s + \beta^*(s)$, where $\alpha^*(s) = \sum_{n=0}^{L-1} \alpha_n s^n$ and $\beta^*(s)$ is a function analytic in a neighborhood of $s = 0$.

(ii) $H^*(s) = \sum_{n=0}^{L-1} (\beta_n - \beta_n^*)s^n +$ higher order terms, where β_n^* is the coefficient of the expansion of $\beta^*(s)$ at $s = 0$, i.e., $\beta^*(s) = \sum_{n=0}^{\infty} \beta_n^* s^n$.

The above (i) and (ii) mean that $\alpha^*(s)$ is equal to the sum of the first L terms of $\alpha(s)$, and the sum of the first L terms of $H^*(s) + \beta^*(s)$ is equal to that of $\beta(s)$.

Define

$$g^*(t) = \sum_{k=0}^{L-1} \frac{g_k^*}{t^{r+k+1}} \Delta(t), \quad t \in \mathbb{R}, \tag{4.26}$$

$$g_k^* = (-1)^{r+k+1}(r + k)!\alpha_k, \quad k = 0, 1, \dots, L - 1, \tag{4.27}$$

where $\Delta(t)$ was defined in (4.1). By Lemma 4.1, we see that $g^*(t)$ satisfies (i). The first coefficient g_0^* is positive by Lemma 4.4, i.e.,

$$g_0^* = (-1)^{r+1}r!\alpha_0 > 0. \tag{4.28}$$

Therefore, we see that $g^*(t)$ is positive for all sufficiently large t.

Let $h_k(t) = k\mathrm{e}^{-kt}$, $t \geqslant 0$, $k = 1, 2, \ldots$, and $H_k(s) = \mathcal{L}(h_k(t))$. We have $H_k(s) = k/(s+k)$, $\Re s > -k$, and the expansion

$$H_k(s) = \sum_{n=0}^{\infty} \Big(-\frac{s}{k}\Big)^n, \quad |s| < k. \tag{4.29}$$

The sum of the first L terms of (4.29) is represented as

$$\sum_{n=0}^{L-1} \Big(-\frac{s}{k}\Big)^n = \Big(1, -\frac{1}{k}, \ldots, \Big(-\frac{1}{k}\Big)^{L-1}\Big)(1, s, \ldots, s^{L-1})^{\mathrm{T}}, \tag{4.30}$$

where $^{\mathrm{T}}$ denotes the transposition of a vector. Let V be the $L \times L$ matrix

$$V = \begin{pmatrix} 1 & -1 & \ldots & (-1)^{L-1} \\ 1 & -\frac{1}{2} & \ldots & \Big(-\frac{1}{2}\Big)^{L-1} \\ \vdots & \vdots & & \vdots \\ 1 & -\frac{1}{L} & \ldots & \Big(-\frac{1}{L}\Big)^{L-1} \end{pmatrix}. \tag{4.31}$$

We will form an objective function $h^*(t)$ by a linear combination of $h_k(t)$, $k = 1, 2, \ldots, L$. Define $h(t) = \sum\limits_{k=1}^{L} d_k h_k(t)$, $t \geqslant 0$, and write $\boldsymbol{d} = (d_1, d_2, \ldots, d_L)$, $\boldsymbol{s} = (1, s, \ldots, s^{L-1})$, and further, $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{L-1})$, $\boldsymbol{\beta^*} = (\beta_0^*, \beta_1^*, \ldots, \beta_{L-1}^*)$. The sum of the first L terms of $H(s) \equiv \mathcal{L}(h(t))$ is $\boldsymbol{d} V \boldsymbol{s}^{\mathrm{T}}$. Then we solve the equation

$$\boldsymbol{d} V \boldsymbol{s}^{\mathrm{T}} = (\boldsymbol{\beta} - \boldsymbol{\beta^*})\boldsymbol{s}^{\mathrm{T}}. \tag{4.32}$$

Since $\det V \neq 0$ (Vandermonde matrix), we have $\boldsymbol{d} = (\boldsymbol{\beta} - \boldsymbol{\beta^*})V^{-1}$. Writing this solution as $\boldsymbol{d} = (d_1, d_2, \ldots, d_L)$, then

$$h^*(t) = \sum_{k=1}^{L} d_k h_k(t) \tag{4.33}$$

is the desired function, i.e., $H^*(s) = \mathcal{L}(h^*(t))$ satisfies (ii).

Summarizing the above, we get:

Lemma 4.6. *Let $\varphi(s)$ be the LS transform of a cumulative distribution function and let its abscissa of convergence be $\sigma_0 = 0$. If*

$$\varphi(s) = \alpha(s)s^r \log s + \beta(s), \quad r \in \mathbb{N}^+, \tag{4.34}$$

where $\alpha(s)$, $\beta(s)$ are analytic in a neighborhood of $s = 0$ with $\alpha(0) \neq 0$, then $g^*(t)$ in (4.26) *and $h^*(t)$ in* (4.33) *satisfy* (i) *and* (ii).

Similarly, in the case of Theorem 3.2, i.e.,

$$\varphi(s) = \alpha(s)s^r + \beta(s), \quad r > 0,\ r \notin \mathbb{N}, \tag{4.35}$$

with $\alpha(s) = \sum_{n=0}^{\infty} \alpha_n s^n$, $\alpha_0 \neq 0$, $\beta(s) = \sum_{n=0}^{\infty} \beta_n s^n$, we will form functions $g^{**}(t)$, $h^{**}(t)$ such that their Laplace transforms $G^{**}(s)$, $H^{**}(s)$ satisfy the following (i′) and (ii′) for $L \in \mathbb{N}^+$.

(i′) $G^{**}(s) = \alpha^{**}(s)s^r + \beta^{**}(s)$, where $\alpha^{**}(s) = \sum_{n=0}^{L-1} \alpha_n s^n$ and $\beta^{**}(s)$ is some function analytic in a neighborhood of $s = 0$.

(ii′) $H^{**}(s) = \sum_{n=0}^{L-1} (\beta_n - \beta_n^{**})s^n +$ higher order terms, where β_n^{**} is the coefficient of the expansion of $\beta^{**}(s)$ at $s = 0$, i.e., $\beta^{**}(s) = \sum_{n=0}^{\infty} \beta_n^{**} s^n$.

In this case, defining

$$g^{**}(t) = \sum_{k=0}^{L-1} \frac{g_k^{**}}{t^{r+k+1}} \Delta(t), \quad t \in \mathbb{R}, \tag{4.36}$$

$$g_k^{**} = (-1)^{r_0+k+1} \frac{\sin \pi\bar{r}}{\pi} \Gamma(r+k+1)\alpha_k, \quad k = 0, 1, \ldots, L-1, \tag{4.37}$$

we see that $g^{**}(t)$ satisfies (i′). The first coefficient g_0^{**} is positive by Lemma 4.5, i.e.,

$$g_0^{**} = (-1)^{r_0+1} \frac{\sin \pi\bar{r}}{\pi} \Gamma(r+1)\alpha_0 > 0.$$

Let $h^{**}(t)$ be a function which is formed similarly to $h^*(t)$ in (4.33) by replacing β_n^* by β_n^{**}. Then $H^{**}(s) = \mathcal{L}(h^{**}(t))$ satisfies (ii′).

Lemma 4.7. *Let $\varphi(s)$ be the LS transform of a cumulative distribution function and let its abscissa of convergence be $\sigma_0 = 0$. If*

$$\varphi(s) = \alpha(s)s^r + \beta(s), \quad r > 0,\ r \notin \mathbb{N}, \tag{4.38}$$

then $g^{**}(t)$ in (4.36) *and $h^{**}(t)$ above satisfy* (i′) *and* (ii′).

4.2. Majorant and minorant functions. For the evaluation of the tail probability $P(X > x)$ from above and from below, we need to use majorant and minorant functions for an exponential function (see Korevaar [7], p. 132, Graham-Vaaler [5]). If two functions f_1, f_2 satisfy $f_1(t) \geqslant f_2(t)$, $t \in \mathbb{R}$, then f_1 is said to be a *majorant* for f_2, and f_2 is a *minorant* for f_1.

For $\omega > 0$, we will define a majorant $M_\omega^1(t)$ and a minorant $m_\omega^1(t)$ for

$$E_\omega(t) \equiv \begin{cases} \mathrm{e}^{-\omega t}, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{4.39}$$

Define (see Korevaar [7], p. 132)

$$M_\omega^1(t) = \Big(\frac{\sin \pi t}{\pi}\Big)^2 Q_\omega(t), \quad t \in \mathbb{R}, \tag{4.40}$$

$$Q_\omega(t) = \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-n\omega}}{(t-n)^2} - \omega \sum_{n=1}^{\infty} \mathrm{e}^{-n\omega}\Big(\frac{1}{t-n} - \frac{1}{t}\Big), \tag{4.41}$$

and

$$m_\omega^1(t) = M_\omega^1(t) - \Big(\frac{\sin \pi t}{\pi t}\Big)^2, \quad t \in \mathbb{R}. \tag{4.42}$$

Moreover, for $L \in \mathbb{N}^+$, define

$$M_\omega^L(t) = (M_\omega^1(t))^L \quad \text{and} \quad m_\omega^L(t) = (m_\omega^1(t))^L. \tag{4.43}$$

For $\sigma > 0$, $\delta > 0$, write $\omega = 2\pi\sigma/\delta$, and then define

$$M_{\sigma,\delta}^L(t) \equiv M_\omega^L\Big(\frac{\delta t}{2\pi}\Big) = M_{2\pi\sigma/\delta}^L\Big(\frac{\delta t}{2\pi}\Big), \tag{4.44}$$

$$m_{\sigma,\delta}^L(t) \equiv m_\omega^L\Big(\frac{\delta t}{2\pi}\Big) = m_{2\pi\sigma/\delta}^L\Big(\frac{\delta t}{2\pi}\Big). \tag{4.45}$$

Lemma 4.8 (Korevaar [7]). *For any $L \in \mathbb{N}^+$, $\sigma > 0$, $\delta > 0$, we have $M^L_{\sigma,\delta}(t)$, $m^L_{\sigma,\delta}(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.*

For $\lambda > 0$, an entire function $f(z)$ of a complex variable $z = x + \mathrm{i}y$ is of *exponential type* λ (see Korevaar [7], p. 128) if

$$|f(z)| \leqslant C \exp(\lambda |z|), \quad z \in \mathbb{C},\ C > 0. \tag{4.46}$$

A real function $f(x)$ is of *type* λ if $f(x)$ is the restriction to $\mathbb{R}$ of an entire function of exponential type λ.

Lemma 4.9 (Korevaar [7], Nakagawa [11]). *For any $L \in \mathbb{N}^+$, $\sigma > 0$, $\delta > 0$, $M^L_{\sigma,\delta}(t)$ and $m^L_{\sigma,\delta}(t)$ are of type $L\delta$.*

Lemma 4.10 (Korevaar [7], Graham-Vaaler [5]). *For $\delta > 0$, $L \in \mathbb{N}^+$,*

$$E_{L\sigma}(t) \leqslant M^L_{\sigma,\delta}(t), \quad t \in \mathbb{R}, \tag{4.47}$$

and for odd $L \in \mathbb{N}^+$,

$$m^L_{\sigma,\delta}(t) \leqslant E_{L\sigma}(t), \quad t \in \mathbb{R}. \tag{4.48}$$

Proof. If $L = 1$, the result follows from Korevaar [7], p. 129, Proposition 5.2. The odd power preserves the order of real numbers, hence (4.48) holds. $\square$

From Lemma 4.8, we can define the Fourier transforms $\widehat{M}^L_{\sigma,\delta} = \mathcal{F}(M^L_{\sigma,\delta})$ and $\hat{m}^L_{\sigma,\delta} = \mathcal{F}(m^L_{\sigma,\delta})$, where the Fourier transform is defined as

$$\widehat{M}^L_{\sigma,\delta}(\tau) = \int_{-\infty}^{\infty} M^L_{\sigma,\delta}(t) \mathrm{e}^{-\mathrm{i}\tau t}\, \mathrm{d}t, \quad \tau \in \mathbb{R}. \tag{4.49}$$

Then, from Lemma 4.9 and the Paley-Wiener theorem [7], [12], we have

Lemma 4.11 (Korevaar [7], Rudin [12]). *For any $L \in \mathbb{N}^+$,*

$$\operatorname{supp}(\widehat{M}^L_{\sigma,\delta}) \subset [-L\delta, L\delta] \quad \text{and} \quad \operatorname{supp}(\hat{m}^L_{\sigma,\delta}) \subset [-L\delta, L\delta], \tag{4.50}$$

where supp *denotes the support of a function.*

4.3. Calculation of $\widehat{M}^L_{\sigma,\delta}(\tau)$ and $\hat{m}^L_{\sigma,\delta}(\tau)$. For $L = 1$, it is not difficult to calculate the Fourier transforms $\widehat{M}^1_\omega = \mathcal{F}(M^1_\omega)$ and $\hat{m}^1_\omega = \mathcal{F}(m^1_\omega)$.

Define

$$q_1(t) = \Big(\frac{\sin \pi t}{\pi t}\Big)^2, \quad q_2(t) = \frac{\sin^2 \pi t}{\pi t}, \quad t \in \mathbb{R}, \tag{4.51}$$

and write $\hat{q}_1 = \mathcal{F}(q_1)$, $\hat{q}_2 = \mathcal{F}(q_2)$. By calculation, we have

$$\hat{q}_1(\tau) = \begin{cases} 1 + \dfrac{\tau}{2\pi}, & -2\pi \leqslant \tau < 0, \\ 1 - \dfrac{\tau}{2\pi}, & 0 \leqslant \tau < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \tag{4.52}$$

and

$$\hat{q}_2(\tau) = \begin{cases} \dfrac{\mathrm{i}}{2}, & -2\pi \leqslant \tau < 0, \\ -\dfrac{\mathrm{i}}{2}, & 0 \leqslant \tau < 2\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{4.53}$$

Lemma 4.12. *We have*

$$\widehat{M}^1_\omega(\tau) = \frac{1}{1 - \mathrm{e}^{-(\omega + \mathrm{i}\tau)}}\hat{q}_1(\tau) - \frac{\omega}{\pi}\Big(\frac{1}{1 - \mathrm{e}^{-(\omega + \mathrm{i}\tau)}} - \frac{1}{1 - \mathrm{e}^{-\omega}}\Big)\hat{q}_2(\tau),$$
$$\hat{m}^1_\omega(\tau) = \frac{\mathrm{e}^{-(\omega + \mathrm{i}\tau)}}{1 - \mathrm{e}^{-(\omega + \mathrm{i}\tau)}}\hat{q}_1(\tau) - \frac{\omega}{\pi}\Big(\frac{1}{1 - \mathrm{e}^{-(\omega + \mathrm{i}\tau)}} - \frac{1}{1 - \mathrm{e}^{-\omega}}\Big)\hat{q}_2(\tau).$$

Proof. See Appendix A. □

Next, we will calculate $\widehat{M}^L_\omega = \mathcal{F}(M^L_\omega)$, $\hat{m}^L_\omega = \mathcal{F}(m^L_\omega)$ and then calculate $\lim\limits_{\omega\to 0+} \widehat{M}^L_\omega(\tau)$, $\lim\limits_{\omega\to 0+} \hat{m}^L_\omega(\tau)$, for $\tau \neq 0$.

Let us define

$$\begin{aligned} u_\omega(t) &= \Big(\frac{\sin \pi t}{\pi t}\Big)^2 - \frac{\omega}{\pi}\frac{\sin^2 \pi t}{\pi t} \\ &= q_1(t) - \frac{\omega}{\pi} q_2(t), \quad t \in \mathbb{R}, \end{aligned} \tag{4.54}$$

$$\begin{aligned} v_\omega(t) &= \frac{\omega}{\pi}\frac{\sin^2 \pi t}{\pi t} \\ &= \frac{\omega}{\pi} q_2(t), \quad t \in \mathbb{R}, \end{aligned} \tag{4.55}$$

and $\hat{u}_\omega = \mathcal{F}(u_\omega)$, $\hat{v}_\omega = \mathcal{F}(v_\omega)$.

Lemma 4.13. *We have*

$$\widehat{M}^L_\omega(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \binom{L}{l} \Big(\frac{1}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}}\Big)^l \Big(\frac{1}{1-\mathrm{e}^{-\omega}}\Big)^{L-l} \hat{u}^{*l}_\omega(\tau) * \hat{v}^{*L-l}_\omega(\tau),$$

$$\hat{m}^L_\omega(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \binom{L}{l} \Big(\frac{\mathrm{e}^{-(\omega+\mathrm{i}\tau)}}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}}\Big)^l \Big(\frac{\mathrm{e}^{-\omega}}{1-\mathrm{e}^{-\omega}}\Big)^{L-l} \hat{u}^{*l}_\omega(\tau) * \hat{v}^{*L-l}_\omega(\tau),$$

where $*$ denotes the convolution operator and $*l$ denotes the l-fold convolutions.

Proof. See Appendix B. □

Lemma 4.14. *For $\tau \neq 0$, we have*

$$\lim_{\omega\to 0+} \widehat{M}^L_\omega(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \Big(\frac{1}{1-\mathrm{e}^{-\mathrm{i}\tau}}\Big)^l \hat{q}_1^{*l}(\tau) * \hat{q}_2^{*L-l}(\tau), \tag{4.56}$$

$$\lim_{\omega\to 0+} \hat{m}^L_\omega(\tau) = \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \Big(\frac{\mathrm{e}^{-\mathrm{i}\tau}}{1-e^{-\mathrm{i}\tau}}\Big)^l \hat{q}_1^{*l}(\tau) * \hat{q}_2^{*L-l}(\tau). \tag{4.57}$$

Proof. Equalities (4.56) and (4.57) follow from

$$\hat{u}^{*l}_\omega(\tau) = \sum_{j=0}^{l} \Big(-\frac{\omega}{\pi}\Big)^{l-j} \hat{q}_1^{*j}(\tau) * \hat{q}_2^{*l-j}(\tau), \quad \hat{v}^{*L-l}_\omega(\tau) = \Big(\frac{\omega}{\pi}\Big)^{L-l} \hat{q}_2^{*L-l}(\tau).$$

□

By the change of variables, we have

Lemma 4.15. *For $\tau \neq 0$,*

$$\begin{aligned}\lim_{\sigma\to 0+} \widehat{M}^L_{\sigma,\delta}(\tau) &= \frac{2\pi}{\delta}\frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \Big(\frac{1}{1-\mathrm{e}^{-\mathrm{i}2\pi\tau/\delta}}\Big)^l \hat{q}_1^{*l}\Big(\frac{2\pi\tau}{\delta}\Big) * \hat{q}_2^{*L-l}\Big(\frac{2\pi\tau}{\delta}\Big)\\ &= \frac{1}{\tau^L} \times (\text{a bounded and piecewise } C^\infty \text{ function}),\\ \lim_{\sigma\to 0+} \hat{m}^L_{\sigma,\delta}(\tau) &= \frac{2\pi}{\delta}\frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \frac{1}{\pi^{L-l}} \binom{L}{l} \Big(\frac{\mathrm{e}^{-\mathrm{i}2\pi\tau/\delta}}{1-\mathrm{e}^{-\mathrm{i}2\pi\tau/\delta}}\Big)^l \hat{q}_1^{*l}\Big(\frac{2\pi\tau}{\delta}\Big) * \hat{q}_2^{*L-l}\Big(\frac{2\pi\tau}{\delta}\Big)\\ &= \frac{1}{\tau^L} \times (\text{a bounded and piecewise } C^\infty \text{ function}).\end{aligned}$$

5. Proof of Theorem 3.1

5.1. Upper bound for $P(X > x)$. First, we will estimate $P(X > x)$ from above by using the majorant function $M_{\sigma,\delta}^{L}$. Let

$$L \geqslant r, \quad L \in \mathbb{N}^{+}. \tag{5.1}$$

For arbitrary $\sigma_1 > 0$, $\sigma_2 > 0$, $\delta > 0$,

$$\begin{aligned} \mathrm{e}^{L\sigma_2 x} \int_x^{\infty} \mathrm{e}^{-(\sigma_1 + L\sigma_2)t} \, \mathrm{d}F(t) &= \int_x^{\infty} \mathrm{e}^{-L\sigma_2(t-x)} \mathrm{e}^{-\sigma_1 t} \, \mathrm{d}F(t) \\ &= \int_0^{\infty} E_{L\sigma_2}(t-x) \mathrm{e}^{-\sigma_1 t} \, \mathrm{d}F(t) \\ &\leqslant \int_0^{\infty} M_{\sigma_2,\delta}^{L}(t-x) \mathrm{e}^{-\sigma_1 t} \, \mathrm{d}F(t), \quad x > 0, \end{aligned} \tag{5.2}$$

where the last inequality holds by (4.47) in Lemma 4.10 (see also Korevaar [7], Nakagawa [11]). By Lemma 4.11, $M_{\sigma_2,\delta}^{L}(t-x)$ is represented by the inverse Fourier transform of $\widehat{M}_{\sigma_2,\delta}^{L} = \mathcal{F}(M_{\sigma_2,\delta}^{L})$ as

$$M_{\sigma_2,\delta}^{L}(t-x) = \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}_{\sigma_2,\delta}^{L}(-\tau) \mathrm{e}^{\mathrm{i}(x-t)\tau} \, \mathrm{d}\tau. \tag{5.3}$$

Substituting (5.3) into (5.2), we have by Fubini's theorem

$$\begin{aligned} \int_0^{\infty} M_{\sigma_2,\delta}^{L}(t-x) \mathrm{e}^{-\sigma_1 t} \, \mathrm{d}F(t) &= \int_0^{\infty} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}_{\sigma_2,\delta}^{L}(-\tau) \mathrm{e}^{\mathrm{i}(x-t)\tau} \mathrm{e}^{-\sigma_1 t} \, \mathrm{d}\tau \, \mathrm{d}F(t) \\ &= \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}_{\sigma_2,\delta}^{L}(-\tau) \mathrm{e}^{\mathrm{i}x\tau} \, \mathrm{d}\tau \int_0^{\infty} \mathrm{e}^{-(\sigma_1 + \mathrm{i}\tau)t} \, \mathrm{d}F(t) \\ &= \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}_{\sigma_2,\delta}^{L}(-\tau) \mathrm{e}^{\mathrm{i}x\tau} \varphi(\sigma_1 + \mathrm{i}\tau) \, \mathrm{d}\tau. \end{aligned} \tag{5.4}$$

Now, let us consider the representation (3.1) of $\varphi(s)$, i.e.,

$$\varphi(s) = \alpha(s) s^{r} \log s + \beta(s), \quad r \in \mathbb{N}^{+}. \tag{5.5}$$

Similarly to the proofs of various complex Tauberian theorems [5], [6], [7], [10], [11], the main idea for the proof of our Theorem 3.1 is to subtract the singular part from the LS transform $\varphi(s)$. If the singularity is a pole, then the singular part, i.e., the principal part of the pole is a finite sum of rational functions. Thus, the inverse Laplace transform of the singular part exists and it is a finite sum of functions.

Contrary to the case of a pole, in this paper, the singular part of $\varphi(s)$ is $\alpha(s)s^r \log s$, which is represented by an infinite series, in general. If we try to make the Laplace inverse of $\alpha(s)s^r \log s$ by force, we have

$$\mathcal{L}^{-1}(\alpha(s)s^r \log s) = \sum_{k=0}^{\infty} \frac{(-1)^{r+k+1}(r+k)!\alpha_k}{t^{r+k+1}} \Delta(t) + \text{some function;} \tag{5.6}$$

however, the first term on the right-hand side of (5.6) is a divergent series, i.e., the radius of convergence is 0, in general. So, we cannot directly apply the technique used in [5], [6], [7], [10], [11] to our theorems. Then, instead of the infinite series in (5.6), we subtract the Laplace transform of a finite series

$$g^*(t) = \sum_{k=0}^{L-1} \frac{(-1)^{r+k+1}(r+k)!\alpha_k}{t^{r+k+1}} \Delta(t) \tag{5.7}$$

from $\varphi(s)$ to weaken the singularity of $\varphi(s)$, where $g^*(t)$ is previously defined in (4.26) and L is an integer that satisfies (5.1).

We define

$$f^*(t) = g^*(t) + h^*(t), \quad t \geqslant 0, \tag{5.8}$$

where $g^*(t)$ and $h^*(t)$ are given in Lemma 4.6. Let $\varphi^*(s) = \mathcal{L}(f^*(t))$, then $f^*(t)$ and $\varphi^*(s)$ have the following properties (a), (b), (c) and (c′):

(a) $f^*(t) > 0$ for all sufficiently large t.

This is because $g^*(0) = g_0^* > 0$ by (4.28).

Define $\xi(s) \equiv \varphi(s) - \varphi^*(s)$, then by Lemma 4.6,

$$\begin{aligned} \xi(s) &= \varphi(s) - (G^*(s) + H^*(s)) \\ &= s^L \left\{ \left(\sum_{n=0}^{\infty} \alpha_{L+n} s^n \right) s^r \log s + \sum_{n=0}^{\infty} (\beta_{L+n} - \beta^*_{L+n}) s^n \right\}, \quad s = \sigma + \mathrm{i}\tau. \end{aligned} \tag{5.9}$$

We have, from (5.9),

(b) $\xi(s)$ is continuous in the closed region $\{0 \leqslant \sigma \leqslant \varepsilon,\ -L\delta \leqslant \tau \leqslant L\delta\} \subset \mathbb{C}$ for sufficiently small $\varepsilon, \delta > 0$.

In fact, because $s = 0$ is an isolated singularity, we can take ε, δ so small that the closed region $\{0 \leqslant \sigma \leqslant \varepsilon,\ -L\delta \leqslant \tau \leqslant L\delta\}$ does not include any singularities of $\varphi(s)$ and $\varphi^*(s)$ other than $s = 0$.

Define $\widehat{M}^L_{0,\delta}(\tau) = \lim_{\sigma\to 0+} \widehat{M}^L_{\sigma,\delta}(\tau)$, $\hat{m}^L_{0,\delta}(\tau) = \lim_{\sigma\to 0+} \hat{m}^L_{\sigma,\delta}(\tau)$, $-L\delta \leqslant \tau \leqslant L\delta$, $\tau \neq 0$, for δ sufficiently small as in (b). Then we have

(c) $\operatorname{supp}(\widehat{M}^L_{0,\delta}) \subset [-L\delta, L\delta]$, and $\widehat{M}^L_{0,\delta}(\tau)\xi(\mathrm{i}\tau)$ is piecewise C^∞ with $(\widehat{M}^L_{0,\delta}(\tau)\xi(\mathrm{i}\tau))^{(r)}$ element of $L^1([-L\delta, L\delta])$.

(c′) $\operatorname{supp}(\hat{m}^L_{0,\delta}) \subset [-L\delta, L\delta]$, and $\hat{m}^L_{0,\delta}(\tau)\xi(\mathrm{i}\tau)$ is piecewise C^∞ with $(\hat{m}^L_{0,\delta}(\tau)\xi(\mathrm{i}\tau))^{(r)}$ element of $L^1([-L\delta, L\delta])$.

Here, ${}^{(r)}$ denotes the r-th derivative. The items (c) and (c′) follow from Lemma 4.15 and (5.9).

Now, defining $F^*(t)$ by $\mathrm{d}F^*(t) = f^*(t)\,\mathrm{d}t$, we have in a way similar to (5.4)

$$\int_0^\infty M^L_{\sigma_2,\delta}(t-x)\mathrm{e}^{-\sigma_1 t}\,\mathrm{d}F^*(t) = \frac{1}{2\pi}\int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\mathrm{e}^{\mathrm{i}x\tau}\varphi^*(\sigma_1+\mathrm{i}\tau)\,\mathrm{d}\tau. \tag{5.10}$$

Subtracting (5.10) from (5.4), we have

$$\begin{aligned}\int_0^\infty & M^L_{\sigma_2,\delta}(t-x)\mathrm{e}^{-\sigma_1 t}\,\mathrm{d}F(t) \\ &= \int_0^\infty M^L_{\sigma_2,\delta}(t-x)\mathrm{e}^{-\sigma_1 t}\,\mathrm{d}F^*(t) + \frac{1}{2\pi}\int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\mathrm{e}^{\mathrm{i}x\tau}\xi(\sigma_1+\mathrm{i}\tau)\,\mathrm{d}\tau.\end{aligned} \tag{5.11}$$

For sufficiently small $\delta > 0$,

$$\xi(\mathrm{i}\tau) = \lim_{\sigma_1\to 0+}\xi(\sigma_1+\mathrm{i}\tau), \quad -L\delta \leqslant \tau \leqslant L\delta, \tag{5.12}$$

is uniform convergence due to (b), hence,

$$\lim_{\sigma_1\to 0+}\frac{1}{2\pi}\int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\xi(\sigma_1+\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau = \frac{1}{2\pi}\int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau. \tag{5.13}$$

From (5.2), (5.11), (5.13), for $\sigma_1 \to 0+$ we have

$$\begin{aligned}\mathrm{e}^{L\sigma_2 x} & \int_x^\infty \mathrm{e}^{-L\sigma_2 t}\,\mathrm{d}F(t) \\ &\leqslant \int_0^\infty M^L_{\sigma_2,\delta}(t-x)\,\mathrm{d}F^*(t) + \frac{1}{2\pi}\int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau.\end{aligned} \tag{5.14}$$

By the estimation for $M^1_\omega(t)$ (see Korevaar [7], p. 132), i.e.,

$$\begin{cases} 0 \leqslant M^1_\omega(t) \leqslant \left(\dfrac{\sin\pi t}{\pi t}\right)^2, & t < 0, \\ \mathrm{e}^{-\omega t} \leqslant M^1_\omega(t) \leqslant \mathrm{e}^{-\omega t} + \left(\dfrac{\sin\pi t}{\pi t}\right)^2, & t \geqslant 0, \end{cases} \tag{5.15}$$

we have

$$
(5.16)\qquad \begin{cases} 0 \leqslant M^L_{\sigma,\delta}(t) \leqslant \Big(\dfrac{\sin \delta t/2}{\delta t/2}\Big)^{2L}, & t < 0, \\ \mathrm{e}^{-L\omega t} \leqslant M^L_{\sigma,\delta}(t) \leqslant \Big(\mathrm{e}^{-\omega t} + \Big(\dfrac{\sin \pi t}{\pi t}\Big)^2\Big)^L, & t \geqslant 0, \end{cases}
$$

where $\omega = 2\pi\sigma/\delta$. Thus, it is easy to see that there exists a constant $C_1 > 0$ such that

$$
(5.17)\qquad M^L_{\sigma,\delta}(t-x) \leqslant \begin{cases} \Big(\dfrac{1}{\delta(x-t)/2}\Big)^{2L}, & 0 \leqslant t < x-1, \\ C_1, & t \geqslant x-1. \end{cases}
$$

Therefore, the first term on the right-hand side of (5.14) is evaluated as

$$
\begin{aligned}
(5.18)\qquad & \int_0^\infty M^L_{\sigma_2,\delta}(t-x)\,\mathrm{d}F^*(t) \\
&= \int_0^\infty M^L_{\sigma_2,\delta}(t-x) f^*(t)\,\mathrm{d}t \\
&= \int_0^\infty M^L_{\sigma_2,\delta}(t-x)(g^*(t)+h^*(t))\,\mathrm{d}t \\
&\leqslant \sum_{k=0}^{L-1} |g^*_k| \int_1^{x-1} \Big(\frac{1}{\delta(x-t)/2}\Big)^{2L} \frac{1}{t^{r+k+1}}\,\mathrm{d}t + C_1 \sum_{k=0}^{L-1} |g^*_k| \int_{x-1}^\infty \frac{1}{t^{r+k+1}}\,\mathrm{d}t \\
&\quad + \sum_{k=1}^{L} k|d_k| \int_0^{x-1} \Big(\frac{1}{\delta(x-t)/2}\Big)^{2L} \mathrm{e}^{-kt}\,\mathrm{d}t + C_1 \sum_{k=1}^{L} k|d_k| \int_{x-1}^\infty \mathrm{e}^{-kt}\,\mathrm{d}t \\
&\leqslant O(x^{-(r+1)}) + C_2 g^*_0 x^{-r} + O(x^{-2L}) + O(\mathrm{e}^{-x}), \quad C_2 > 0 \\
&< \frac{C_3}{x^r}, \quad C_3 > 0,
\end{aligned}
$$

for all sufficiently large x, by virtue of Lemmas Appendix C.1, Appendix C.2 in Appendix C. Note that $g^*_0 > 0$ and $L \geqslant r$.

Next, the second term on the right-hand side of (5.14) will be estimated. We have by (c) and integration by parts,

$$
\begin{aligned}
(5.19)\qquad \lim_{\sigma_2 \to 0+} \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}^L_{\sigma_2,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau &= \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \widehat{M}^L_{0,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau \\
&= \frac{\mathrm{i}^r}{2\pi x^r} \int_{-L\delta}^{L\delta} (\widehat{M}^L_{0,\delta}(-\tau)\xi(\mathrm{i}\tau))^{(r)} \mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau \\
&= o(x^{-r}), \quad x \to \infty,
\end{aligned}
$$

due to the Riemann-Lebesgue theorem. Then in (5.14) for $\sigma_2 \to 0+$, we have by (5.18) and (5.19),

$$P(X > x) = \int_x^\infty \mathrm{d}F(t) < \frac{C}{x^r}, \quad C > 0, \tag{5.20}$$

for all sufficiently large x.

5.2. Lower bound for $P(X > x)$. We will estimate $P(X > x)$ from below by using the minorant function $m_{\sigma,\delta}^L$. Let $L \in \mathbb{N}^+$ be an odd number with $L \geqslant r$. For arbitrary $\sigma_1 > 0$, $\sigma_2 > 0$, $\delta > 0$,

$$\begin{aligned} \mathrm{e}^{L\sigma_2 x} \int_x^\infty \mathrm{e}^{-(\sigma_1 + L\sigma_2)t} \,\mathrm{d}F(t) &= \int_0^\infty E_{L\sigma_2}(t-x)\mathrm{e}^{-\sigma_1 t} \,\mathrm{d}F(t) \\ &\geqslant \int_0^\infty m_{\sigma_2,\delta}^L(t-x)\mathrm{e}^{-\sigma_1 t} \,\mathrm{d}F(t), \quad x > 0. \end{aligned} \tag{5.21}$$

In a way similar to that from (5.2) to (5.14), we have

$$\begin{aligned} &\mathrm{e}^{L\sigma_2 x} \int_x^\infty \mathrm{e}^{-L\sigma_2 t} \,\mathrm{d}F(t) \\ &\quad \geqslant \int_0^\infty m_{\sigma_2,\delta}^L(t-x) \,\mathrm{d}F^*(t) + \frac{1}{2\pi} \int_{-L\delta}^{L\delta} \hat{m}_{\sigma_2,\delta}^L(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau} \,\mathrm{d}\tau. \end{aligned} \tag{5.22}$$

By the estimation for $m_\omega^1(t)$ (see Korevaar [7], p. 132), i.e.,

$$\begin{cases} -\Big(\dfrac{\sin \pi t}{\pi t}\Big)^2 \leqslant m_\omega^1(t) \leqslant 0, & t < 0, \\ \mathrm{e}^{-\omega t} - \Big(\dfrac{\sin \pi t}{\pi t}\Big)^2 \leqslant m_\omega^1(t) \leqslant \mathrm{e}^{-\omega t}, & t \geqslant 0, \end{cases} \tag{5.23}$$

we have

$$\begin{cases} -\Big(\dfrac{\sin \delta t/2}{\delta t/2}\Big)^{2L} \leqslant m_{\sigma,\delta}^L(t) \leqslant 0, & t < 0, \\ \Big(\mathrm{e}^{-\omega t} - \Big(\dfrac{\sin \delta t/2}{\delta t/2}\Big)^2\Big)^L \leqslant m_{\sigma,\delta}^L(t) \leqslant \mathrm{e}^{-L\omega t}, & t \geqslant 0, \end{cases} \tag{5.24}$$

where $\omega = 2\pi\sigma/\delta$. Thus, there exist constants $C_4, C_5, C_6 > 0$ such that

$$m_{\sigma,\delta}^L(t-x) \geqslant \begin{cases} -\Big(\dfrac{1}{\delta(x-t)/2}\Big)^{2L}, & 0 \leqslant t < x-1, \\ -C_4, & x-1 \leqslant t < x+1, \\ C_5\mathrm{e}^{-(2\pi L\sigma/\delta)(t-x)} - C_6\Big(\dfrac{1}{\delta(x-t)/2}\Big)^2, & t \geqslant x+1. \end{cases}$$

Therefore, the first term on the right-hand side of (5.22) is estimated as

$$
\begin{aligned}
&\int_0^\infty m^L_{\sigma_2,\delta}(t-x)(g^*(t)+h^*(t))\,\mathrm{d}t \\
&\quad \geqslant -\sum_{k=0}^{L-1}|g_k^*|\int_1^{x-1}\Big(\frac{1}{\delta(x-t)/2}\Big)^{2L}\frac{1}{t^{r+k+1}}\,\mathrm{d}t - C_4\sum_{k=0}^{L-1}|g_k^*|\int_{x-1}^{x+1}\frac{1}{t^{r+k+1}}\,\mathrm{d}t \\
&\qquad + C_5\sum_{k=0}^{L-1}|g_k^*|\int_{x+1}^{\infty}\mathrm{e}^{-(2\pi L\sigma_2/\delta)(t-x)}\frac{1}{t^{r+k+1}}\,\mathrm{d}t \\
&\qquad - C_6\sum_{k=0}^{L-1}|g_k^*|\int_{x+1}^{\infty}\Big(\frac{1}{\delta(x-t)/2}\Big)^{2}\frac{1}{t^{r+k+1}}\,\mathrm{d}t \\
&\qquad - \sum_{k=1}^{L}k|d_k|\int_0^{x-1}\Big(\frac{1}{\delta(x-t)/2}\Big)^{2L}\mathrm{e}^{-kt}\,\mathrm{d}t - C_4\sum_{k=1}^{L}k|d_k|\int_{x-1}^{x+1}\mathrm{e}^{-kt}\,\mathrm{d}t \\
&\qquad + C_5\sum_{k=1}^{L}k|d_k|\int_{x+1}^{\infty}\mathrm{e}^{-(2\pi L\sigma/\delta)(t-x)}\mathrm{e}^{-kt}\,\mathrm{d}t \\
&\qquad - C_6\sum_{k=1}^{L}k|d_k|\int_{x+1}^{\infty}\Big(\frac{1}{\delta(x-t)/2}\Big)^{2}\mathrm{e}^{-kt}\,\mathrm{d}t \\
&\quad \geqslant O(x^{-(r+1)}) + O(x^{-(r+1)}) + C_5' g_0^*\int_{x+1}^{\infty}\mathrm{e}^{-(2\pi L\sigma_2/\delta)(t-x)}\frac{1}{t^{r+1}}\,\mathrm{d}t \\
&\qquad + O(x^{-(r+1)}) + O(x^{-2L}) + O(\mathrm{e}^{-x}) + O(\mathrm{e}^{-x}) + O(\mathrm{e}^{-x}),
\end{aligned}
$$

where C_5' is a positive constant. Thus, we have, from $g_0^*>0$,

$$
\lim_{\sigma_2\to 0+}\int_0^\infty m^L_{\sigma_2,\delta}(t-x)\,\mathrm{d}F^*(t) \geqslant \frac{C_7}{x^r}, \quad C_7>0, \tag{5.25}
$$

for all sufficiently large x.

Next, we will evaluate the second term in the right-hand side of (5.22). We have, by (c′) and the integration by parts,

$$
\begin{aligned}
\lim_{\sigma_2\to 0+}\frac{1}{2\pi}\int_{-L\delta}^{L\delta}\hat{m}^L_{\sigma_2,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau &= \frac{1}{2\pi}\int_{-L\delta}^{L\delta}\hat{m}^L_{0,\delta}(-\tau)\xi(\mathrm{i}\tau)\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau \\
&= \frac{\mathrm{i}^r}{2\pi x^r}\int_{-L\delta}^{L\delta}(\hat{m}^L_{0,\delta}(-\tau)\xi(\mathrm{i}\tau))^{(r)}\mathrm{e}^{\mathrm{i}x\tau}\,\mathrm{d}\tau \\
&= o(x^{-r}), \quad x\to\infty,
\end{aligned} \tag{5.26}
$$

due to the Riemann-Lebesgue theorem. Then, using (5.22) for $\sigma_2 \to 0+$, we have from (5.25) and (5.26),

$$P(X > x) = \int_x^{\infty} \mathrm{d}F(t) > \frac{C'}{x^r}, \quad C' > 0, \tag{5.27}$$

for all sufficiently large x.

By (5.20) and (5.27), the proof of Theorem 3.1 is completed. $\square$

6. Proof of Theorem 3.2

The same proof as that of Theorem 3.1 is applicable to Theorem 3.2 by replacing $g^*(t)$, $h^*(t)$ in Lemma 4.6 with $g^{**}(t)$, $h^{**}(t)$ in Lemma 4.7. $\square$

7. Application to the stationary distribution of M/G/1 type Markov chain

We will apply our theorems to the tail probability of the stationary distribution of an M/G/1 type Markov chain [3].

Let us consider an irreducible and positive recurrent M/G/1 type Markov chain with the state transition probability matrix

$$P = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{7.1}$$

where $A \equiv (a_n)_{n\in\mathbb{N}}$, $B \equiv (b_n)_{n\in\mathbb{N}}$ are probability vectors. Let us denote by X_A, X_B the random variables with probability distribution A, B, respectively, that is, $P(X_A = n) = a_n$, $P(X_B = n) = b_n$, $n \in \mathbb{N}$. Denote by $A(z)$, $B(z)$ the probability generating functions of X_A, X_B, respectively, i.e., $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $B(z) = \sum_{n=0}^{\infty} b_n z^n$, and then define $\varphi_A(s) = A(\mathrm{e}^{-s})$, $\varphi_B(s) = B(\mathrm{e}^{-s})$ by substituting $z = \mathrm{e}^{-s}$. Further, denote by $E(X_A)$, $E(X_B)$ the expectations of X_A, X_B, respectively, i.e., $E(X_A) = \sum_{n=1}^{\infty} n a_n$, $E(X_B) = \sum_{n=1}^{\infty} n b_n$. It is known [3] that if P is positive recurrent, then $E(X_A) < 1$ and $E(X_B) < \infty$.

Denote by $\pi = (\pi_n)_{n\in\mathbb{N}}$ the stationary distribution of P, i.e., $\pi = \pi P$, and write its probability generating function as $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$. Then denote by X_π the random variable defined by $P(X_\pi = n) = \pi_n$, $n \in \mathbb{N}$.

We have the following relation [3]:

$$\pi(z) = \frac{\pi_0(zB(z) - A(z))}{z - A(z)}. \tag{7.2}$$

Having a power series expansion of the right-hand side of (7.2), we can obtain π_n, however, in general, this is difficult or almost impossible. Our purpose is to know the asymptotic decay of the tail probability $P(X_\pi > n)$, so we do not need to obtain the exact value of π_n.

Now, we apply our theorems to this problem. We assume the following conditions (I), (II), (III).

(I) X_B is of Pareto type and $\varphi_B(s)$ has a singularity of the form as in Theorem 3.1:

$$\varphi_B(s) = \alpha(s)s^r \log s + \beta(s), \tag{7.3}$$

where $r \in \mathbb{N}$, $r \geqslant 2$ and $\alpha(s)$, $\beta(s)$ are analytic in a neighborhood of $s = 0$ with $\alpha(0) \neq 0$, or of the form as in Theorem 3.2:

$$\varphi_B(s) = \alpha(s)s^r + \beta(s), \tag{7.4}$$

where $r > 1$, $r \notin \mathbb{N}$ and $\alpha(s)$, $\beta(s)$ are analytic in a neighborhood of $s = 0$ with $\alpha(0) \neq 0$.

(II) The radius of convergence r_A of $A(z)$ is greater than 1, i.e., $r_A > 1$.

(III) $z = 1$ is a simple zero of $z - A(z)$, i.e., we have $z - A(z) = (z - 1)u(z)$ with $u(1) \neq 0$.

Remark 7.1. Due to (I), the abscissa of convergence σ_0 of $\varphi_B(s)$ is $\sigma_0 = 0$, hence by the correspondence $z = \mathrm{e}^{-s}$, the radius of convergence r_B of $B(z)$ is $r_B = 1$.

Remark 7.2. One example of (7.3) is given in (2.3). In (2.3), if $r \geqslant 2$, $r \in \mathbb{N}$, then the expectation satisfies $E(X_B) < \infty$.

Under the assumptions (I), (II), (III), let us first consider the case (7.3). From (7.2) and (III), we have

$$\begin{aligned} \varphi_\pi(s) &= \pi_0 \frac{\mathrm{e}^{-s}\varphi_B(s) - \varphi_A(s)}{(\mathrm{e}^{-s} - 1)u(\mathrm{e}^{-s})} \\ &= \tilde{\alpha}(s)s^{r-1} \log s + \tilde{\beta}(s), \end{aligned} \tag{7.5}$$

where

$$\tilde{\alpha}(s) = \pi_0 \frac{s\mathrm{e}^{-s}\alpha(s)}{(\mathrm{e}^{-s}-1)u(\mathrm{e}^{-s})}, \ \tilde{\beta}(s) = \pi_0 \frac{\mathrm{e}^{-s}\beta(s) - \varphi_A(s)}{(\mathrm{e}^{-s}-1)u(\mathrm{e}^{-s})}. \tag{7.6}$$

Because $\alpha(0) \neq 0$ and $u(1) \neq 0$, we see that $\tilde{\alpha}(s)$ is analytic in a neighborhood of $s = 0$ with $\tilde{\alpha}(0) \neq 0$. Furthermore, because $\beta(0) = 1$ and $\varphi_A(0) = 1$, we see that $\tilde{\beta}(s)$ is analytic in a neighborhood of $s = 0$. Therefore, we can apply our Theorem 3.1 to see that the stationary distribution π of P is of Pareto type.

In a similar way, for (7.4) we have

$$\varphi_\pi(s) = \tilde{\alpha}(s)s^{r-1} + \tilde{\beta}(s), \tag{7.7}$$

where $\tilde{\alpha}(s)$, $\tilde{\beta}(s)$ are the same as in (7.6). Applying Theorem 3.2, we see also in this case that π is of Pareto type.

8. Conclusion

In this paper we investigated the asymptotic decay of the tail probability of a Pareto type random variable. We proved two theorems which give sufficient conditions for a random variable to be of Pareto type. Our theorems are based on the Tauberian theorems by Graham and Vaarler.

We applied our theorems to the tail probability of the stationary distribution π of an M/G/1 type Markov chain. By investigating the singularity of the algebraic representation of $\pi(z)$ with $A(z)$, $B(z)$, we obtained a simple criterion whether π is of Pareto type.

Appendix A. Proof of Lemma 4.12

We have from (4.51)

$$\begin{aligned}
&\widehat{M^1_\omega}(\tau) \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-n\omega} \mathcal{F}\Big(\Big(\frac{\sin \pi(t-n)}{\pi(t-n)}\Big)^2\Big) - \frac{\omega}{\pi} \sum_{n=0}^{\infty} \mathrm{e}^{-n\omega} \Big\{\mathcal{F}\Big(\frac{\sin^2 \pi(t-n)}{\pi(t-n)}\Big) - \mathcal{F}\Big(\frac{\sin^2 \pi t}{\pi t}\Big)\Big\} \\
&= \hat{q}_1(\tau) \sum_{n=0}^{\infty} \mathrm{e}^{-n(\omega+\mathrm{i}\tau)} - \frac{\omega}{\pi} \hat{q}_2(\tau) \Big(\sum_{n=0}^{\infty} \mathrm{e}^{-n(\omega+\mathrm{i}\tau)} - \sum_{n=0}^{\infty} \mathrm{e}^{-n\omega}\Big) \\
&= \frac{1}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}} \hat{q}_1(\tau) - \frac{\omega}{\pi}\Big(\frac{1}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}} - \frac{1}{1-\mathrm{e}^{-\omega}}\Big)\hat{q}_2(\tau).
\end{aligned}$$

The result for $\hat{m}^1_\omega$ is proved in a similar way. $\square$

We have

$$
\begin{aligned}
M_\omega^L(t) &= \Big(\frac{\sin \pi t}{\pi t}\Big)^{2L} (Q_\omega(t))^L \\
&= \Big(\frac{\sin \pi t}{\pi t}\Big)^{2L} \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \Big\{\frac{1}{(t-n_1)^2} - \frac{\omega}{t-n_1} + \frac{\omega}{t}\Big\} \times \dots \\
&\quad \times \Big\{\frac{1}{(t-n_L)^2} - \frac{\omega}{t-n_L} + \frac{\omega}{t}\Big\} \\
&= \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \Big\{\Big(\frac{\sin \pi(t-n_1)}{\pi(t-n_1)}\Big)^2 - \frac{\omega}{\pi}\frac{\sin^2 \pi(t-n_1)}{\pi(t-n_1)} + \frac{\omega}{\pi}\frac{\sin^2 \pi t}{\pi t}\Big\} \times \dots \\
&\quad \times \Big\{\Big(\frac{\sin \pi(t-n_L)}{\pi(t-n_L)}\Big)^2 - \frac{\omega}{\pi}\frac{\sin^2 \pi(t-n_L)}{\pi(t-n_L)} + \frac{\omega}{\pi}\frac{\sin^2 \pi t}{\pi t}\Big\} \\
&= \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \{u_\omega(t-n_1) + v_\omega(t)\} \times \dots \times \{u_\omega(t-n_L) + v_\omega(t)\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\widehat{M}_\omega^L(\tau) = \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \frac{1}{(2\pi)^{L-1}} \{\mathrm{e}^{-\mathrm{i}n_1\tau}\hat{u}_\omega(\tau) + \hat{v}_\omega(\tau)\} * \dots \\
* \{\mathrm{e}^{-\mathrm{i}n_L\tau}\hat{u}_\omega(\tau) + \hat{v}_\omega(\tau)\}
\end{aligned}
$$

(Appendix B.1)

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{L-1}} \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \\
&\quad \times \sum_{l=0}^{L} \Bigg\{ \sum_{k_1,\dots,k_l:\ \text{distinct}} (\mathrm{e}^{-\mathrm{i}n_{k_1}\tau}\hat{u}_\omega(\tau)) * \dots * (\mathrm{e}^{-\mathrm{i}n_{k_l}\tau}\hat{u}_\omega(\tau)) \Bigg\} * \hat{v}_\omega^{*L-l}(\tau);
\end{aligned}
$$

(Appendix B.2)

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{L-1}} \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \sum_{l=0}^{L} \Bigg(\sum_{k_1,\dots,k_l:\ \text{distinct}} \mathrm{e}^{-\mathrm{i}n_{k_1}\tau} \dots \mathrm{e}^{-\mathrm{i}n_{k_l}\tau} \Bigg) \\
&\quad \times \hat{u}_\omega^{*l}(\tau) * \hat{v}_\omega^{*L-l}(\tau);
\end{aligned}
$$

(Appendix B.3)

$$
\begin{aligned}
&= \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \hat{u}_{\omega}^{*l}(\tau) * \hat{v}_{\omega}^{*L-l}(\tau) \sum_{n_1=0}^{\infty} \dots \sum_{n_L=0}^{\infty} \mathrm{e}^{-n_1\omega} \dots \mathrm{e}^{-n_L\omega} \\
&\quad \times \Bigg(\sum_{k_1,\dots,k_l\colon \text{ distinct}} \mathrm{e}^{-\mathrm{i}n_{k_1}\tau} \dots \mathrm{e}^{-\mathrm{i}n_{k_l}\tau} \Bigg) \\
&= \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \hat{u}_{\omega}^{*l}(\tau) * \hat{v}_{\omega}^{*L-l}(\tau) \binom{L}{l} \Big(\frac{1}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}} \Big)^{l} \Big(\frac{1}{1-\mathrm{e}^{-\omega}} \Big)^{L-l} \\
&= \frac{1}{(2\pi)^{L-1}} \sum_{l=0}^{L} \binom{L}{l} \Big(\frac{1}{1-\mathrm{e}^{-(\omega+\mathrm{i}\tau)}} \Big)^{l} \Big(\frac{1}{1-\mathrm{e}^{-\omega}} \Big)^{L-l} \hat{u}_{\omega}^{*l}(\tau) * \hat{v}_{\omega}^{*L-l}(\tau).
\end{aligned}
$$

In (Appendix B.1), (Appendix B.2), (Appendix B.3), for $l = 0$, the (empty) sum on $k_1, \dots, k_l$ is considered to be 1.

Similarly, we have the result for $\hat{m}_{\omega}^{L}(\tau)$. □

Appendix C. Auxiliary Lemmas

Lemma Appendix C.1. *For $n_1, n_2 \in \mathbb{N}^+$, $n_1 \geqslant 2$, $n_2 \geqslant 2$, let $n = \min(n_1, n_2)$. Then*

$$\int_1^{x-1} \frac{\mathrm{d}t}{(x-t)^{n_1} t^{n_2}} \leqslant O(x^{-n}), \quad x \to \infty.$$

Proof. By the change of variables $t = xu$,

$$
\begin{aligned}
\int_1^{x-1} \frac{\mathrm{d}t}{(x-t)^{n_1} t^{n_2}} &= \frac{1}{x^{n_1+n_2+1}} \int_{1/x}^{1-1/x} \frac{\mathrm{d}u}{(1-u)^{n_1} u^{n_2}} \\
&= \frac{1}{x^{n_1+n_2+1}} \Big\{ \int_{1/x}^{1/2} + \int_{1/2}^{1-1/x} \Big\} \frac{\mathrm{d}u}{(1-u)^{n_1} u^{n_2}} \\
&\leqslant \frac{1}{x^{n_1+n_2+1}} \Big\{ 2^{n_1} \int_{1/x}^{1/2} \frac{\mathrm{d}u}{u^{n_2}} + 2^{n_2} \int_{1/2}^{1-1/x} \frac{\mathrm{d}u}{(1-u)^{n_1}} \Big\} \\
&= \frac{1}{x^{n_1+n_2+1}} \Big\{ 2^{n_1} \frac{x^{n_2-1} - 2^{n_2-1}}{n_2 - 1} + 2^{n_2} \frac{x^{n_1-1} - 2^{n_1-1}}{n_1 - 1} \Big\} \\
&= O(x^{-n}), \quad x \to \infty.
\end{aligned}
$$

□

Lemma Appendix C.2. *For $k > 0$ and $n \in \mathbb{N}$, we have*

$$\int_0^{x-1} \frac{\mathrm{e}^{-kt}}{(x-t)^n}\,\mathrm{d}t = O(x^{-n}), \quad x \to \infty.$$

P r o o f. By the change of variables $u = x - t$,

$$\begin{aligned}\int_0^{x-1} \frac{\mathrm{e}^{-kt}}{(x-t)^n}\,\mathrm{d}t &= \mathrm{e}^{-kx} \int_1^x \frac{\mathrm{e}^k u}{u^n}\,\mathrm{d}u \\ &= \frac{1}{x^n} \int_1^x \frac{\mathrm{e}^{ku}}{u^n}\,\mathrm{d}u \Big/ \frac{\mathrm{e}^{kx}}{x^n} \\ &\to \frac{1}{kx^n}, \quad x \to \infty,\end{aligned}$$

from L'Hopital's rule. □

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