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## On the Exponential Decay Rate of the Tail of a Discrete Probability Distribution

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### ABSTRACT

We give a sufficient condition for the exponential decay of the tail of a discrete probability distribution  $\pi = (\pi_n)_{n \geq 0}$  in the sense that  $\lim_{n \rightarrow \infty} (1/n) \log \sum_{i > n} \pi_i = -\theta$  with  $0 < \theta < \infty$ . We focus on analytic properties of the probability generating function of a discrete probability distribution, especially, the radius of convergence and the number of poles on the circle of convergence. Furthermore, we give an example of an M/G/1 type Markov chain such that the tail of its stationary distribution does not decay exponentially.

*Key Words:* Cauchy–Hadamard’s theorem; Complex function theory; Exponential decay rate; Markov chain; Tail of distribution.

### 1. INTRODUCTION

Let us consider a discrete probability distribution  $\pi = (\pi_n)_{n \geq 0}$  and define the tail of the  $\pi$  by  $\Pi_n \equiv \sum_{i > n} \pi_i$ . If  $\Pi_n$  decreases as

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n = -\theta, \quad 0 < \theta < \infty, \tag{1}$$

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1 then we say that the tail  $\Pi_n$  decays exponentially and call the  $\theta$  the exponential  
 2 decay rate of  $\Pi_n$ . In this paper, we investigate conditions for the exponential decay  
 3 of  $\Pi_n$  and determine the exponential decay rate  $\theta$ .

4 This line of active research has been motivated by the need to evaluate a cell  
 5 loss probability as small as  $10^{-10}$  in ATM networks. In Ref.<sup>[1]</sup>, based on large  
 6 deviation theory, for the stationary queue length  $Q$ , an upper bound of the form  
 7  $P(Q > n) \leq \psi \exp(-\theta n)$  is obtained, where  $\psi$  is a constant called an asymptotic  
 8 constant. When a large deviation condition is satisfied, it is shown that  $P(Q > n)$   
 9 decays as  $\lim_{n \rightarrow \infty} P(Q > n) \exp(\theta n) = \psi$ . In Refs.<sup>[6,7]</sup>, the stationary distribution of  
 10 M/G/1 or G/M/1 type Markov chains are deeply studied. In Ref.<sup>[9]</sup>, the tail of  
 11 the waiting time in PH/PH/c queue is investigated. In Ref.<sup>[2]</sup>, sufficient conditions  
 12 are given for the stationary distribution  $\pi = (\pi_n)$  of an M/G/1 type Markov chain  
 13 with boundary modification to decay as  $\pi_n \simeq Kr^{-n} + O(\tilde{r}^{-n})$ ,  $K > 0$ ,  $0 < r < \tilde{r}$ . The  
 14 result is applied to MAP/G/1 queues.

15 In this paper, we first consider a sequence of complex numbers, or a complex  
 16 sequence, for short. We define the tail of a complex sequence and give a sufficient  
 17 condition for the exponential decay of the tail of the complex sequence. We focus  
 18 on analytic properties of the power series whose coefficients are the given complex  
 19 sequence such as the radius of convergence and the number of poles on the circle  
 20 of convergence. The results are of course applicable to discrete probability distribu-  
 21 tions. We give a counter example for the Proposition 1 in Ref.<sup>[3]</sup>, which claims a  
 22 stronger result than in this paper. Furthermore, we give an example of an M/G/1  
 23 type Markov chain such that the tail of its stationary distribution does not decay  
 24 exponentially.  
 25

## 27 2. ASYMPTOTIC BEHAVIOR OF THE TAIL OF 28 DISCRETE PROBABILITY DISTRIBUTION

### 30 2.1. Cauchy–Hadamard’s Theorem

31 In general, for a complex sequence  $c = (c_n)_{n \geq 0}$ , let us consider a power series  
 32

$$33 f(z) = \sum_{n=0}^{\infty} c_n z^n \quad (2)$$

34 and study the asymptotic behavior of  $c = (c_n)$  according to analytic properties of  
 35  $f(z)$ . The basic tool for this purpose is Cauchy–Hadamard’s theorem on the radius  
 36 of convergence of a power series.  
 37

38 **Theorem** (Cauchy–Hadamard,<sup>[5]</sup> p. 344). *Denote by  $r$  the radius of convergence of*  
 39 *a power series (2). Then, we have*  
 40

$$41 \limsup_{n \rightarrow \infty} |c_n|^{1/n} = r^{-1}. \quad (3)$$

1 For a discrete probability distribution  $\pi = (\pi_n)$ ,  $\pi_n \geq 0$ ,  $n = 0, 1, \dots, \sum_{n=0}^{\infty} \pi_n = 1$ , the probability generating function  $\pi(z)$  of  $\pi$  is defined as a power series;

$$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \pi(z) = \sum_{n=0}^{\infty} \pi_n z^n. \quad (4)$$

7 Since  $\pi(1) = 1$ , the radius of convergence  $r$  of  $\pi(z)$  satisfies  $r \geq 1$ . By Cauchy–Hadamard’s theorem, we have

$$8 \quad 9 \quad 10 \quad 11 \quad 12 \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \pi_n = -\log r. \quad (5)$$

13 In this paper, we give a condition for the exponential decay of the tail of  $\pi$ . We will show in Theorem 3 that if the radius of convergence  $r$  of  $\pi(z)$  satisfies  $1 < r < \infty$  and  $\pi(z)$  has a finite number of poles on the circle of convergence  $|z| = r$  and is holomorphic on some neighborhood of  $|z| \leq r$  except for the poles, then the tail of  $\pi$  decays exponentially. These conditions hold if  $\pi(z)$  is meromorphic on some neighborhood of  $|z| \leq r$ , so we can say that in almost all realistic cases we have the limit of the form (1).

## 2.2. Asymptotic Behavior of Complex Sequence

22 In this section, we investigate the asymptotic behavior of a complex sequence  $c = (c_n)$ , which is not necessarily a probability vector. The tail  $C_n$  of  $c$  is defined by  $C_n \equiv \sum_{i>n} |c_i|$ . If the limit

$$23 \quad 24 \quad 25 \quad 26 \quad 27 \quad 28 \quad 29 \quad 30 \quad 31 \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log C_n = -\theta \quad (6)$$

32 exists and  $0 < \theta < \infty$  holds, then we say that the tail  $C_n$  decays exponentially and call the  $\theta$  the exponential decay rate of  $C_n$ . To investigate the asymptotic behavior of  $C_n$ , we focus on analytic properties of the power series  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  with coefficients  $c = (c_n)$ .

33 In general, if a function  $f(z)$  is holomorphic on an open set  $U$  except for poles, then  $f(z)$  is said to be *meromorphic* on  $U$ . For a function  $f(z)$ , if there exists  $\delta > 0$  such that the  $f(z)$  is meromorphic on  $|z| < r + \delta$ , then we say that  $f(z)$  is meromorphic on  $|z| \leq r$  for the sake of simplicity. Likewise, if there exists  $\delta > 0$  such that  $f(z)$  is holomorphic on  $|z| < r + \delta$ , then we say that  $f(z)$  is holomorphic on  $|z| \leq r$ . We have the next lemma.

34 **Lemma 1.** *Let  $r$  be the radius of convergence of a power series  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  with  $0 < r < \infty$ . Then,  $f(z)$  is meromorphic on  $|z| \leq r$  if and only if the singularities of  $f(z)$  on the circle of convergence  $|z| = r$  are only a finite number of poles.*

1 For a power series  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  with the radius of convergence  $r$ , from  
 2 Cauchy–Hadamard’s theorem, we see that there exists a subsequence  $\{n_k\}$  with  
 3  $\lim_{k \rightarrow \infty} |c_{n_k}|^{1/n_k} = r^{-1}$ . We investigate the asymptotic behavior of  $c = (c_n)$  by  
 4 considering this subsequence  $\{n_k\}$ .

5 Denote by  $\mathbb{N}$  the set of natural numbers, i.e.,  $\mathbb{N} = \{0, 1, \dots\}$  and  $\mathbb{N}^+$  the set of  
 6 positive natural numbers.

7  
 8 **Theorem 1.** *Let  $r$  denote the radius of convergence of a power series  $f(z) = \sum_{n=0}^{\infty} c_n z^n$   
 9 with  $0 < r < \infty$ . Assume that  $f(z)$  is meromorphic on  $|z| \leq r$ . Let  $m$  denote the number  
 10 of poles of  $f(z)$  on  $|z| = r$ . Then there exist  $m^* \in \mathbb{N}^+$ ,  $1 \leq m^* \leq m$ , and a sequence  
 11  $\{n_k\}_{k \geq 0} \subset \mathbb{N}$  which satisfy*

$$12 \quad \lim_{k \rightarrow \infty} |c_{n_k}|^{1/n_k} = r^{-1}, \quad (7)$$

$$13 \quad km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \dots \quad (8)$$

14  
 15  
 16 *Proof.* See Appendix.

17  
 18 **Remark 1.** From the proof of Theorem 1, we see that  $m^*$  is the number of poles  
 19 with the largest order. Hence from Theorem 1, if  $m^* = 1$ , then  $\lim_{n \rightarrow \infty} |c_n|^{1/n} = r^{-1}$   
 20 holds.  
 21

22 From Theorem 1, we have

23  
 24 **Theorem 2.** *Let  $r$  be the radius of convergence of a power series  $f(z) = \sum_{n=0}^{\infty} c_n z^n$ ,  
 25 and define  $C_n = \sum_{i>n} |c_i|$ . If  $1 < r < \infty$  and  $f(z)$  is meromorphic on  $|z| \leq r$ , then  
 26 we have*

$$27 \quad \lim_{n \rightarrow \infty} C_n^{1/n} = r^{-1}. \quad (9)$$

28  
 29 *Proof.* First we show

$$30 \quad \limsup_{n \rightarrow \infty} C_n^{1/n} \leq r^{-1}. \quad (10)$$

31  
 32 In fact, by Cauchy–Hadamard’s theorem, for any  $\epsilon$ ,  $0 < \epsilon < 1 - r^{-1}$ , there  
 33 exists  $n_0 \in \mathbb{N}$  such that  $|c_n|^{1/n} \leq r^{-1} + \epsilon$  holds for any  $n \geq n_0$ . Thus, we have  $C_n \leq$   
 34  $(r^{-1} + \epsilon)^{n+1} / (1 - r^{-1} - \epsilon)$  which leads to (10).

35 Next, we show

$$36 \quad \liminf_{n \rightarrow \infty} C_n^{1/n} \geq r^{-1}. \quad (11)$$

37 Let  $m$  be the number of poles of  $f(z)$  on  $|z| = r$ . Then, by Theorem 1, there  
 38 exists  $m^* \in \mathbb{N}^+$ ,  $1 \leq m^* \leq m$ , such that for any  $\epsilon$ ,  $0 < \epsilon < r^{-1}$ , and any  $n \geq n_0$ ,  
 39 there is at least one, say,  $c_{i_0}$ , among  $c_{n+1}, \dots, c_{n+m^*}$  that satisfies  $|c_{i_0}|^{1/i_0} \geq r^{-1} - \epsilon$ .  
 40 Therefore,  $C_n \geq |c_{i_0}| \geq (r^{-1} - \epsilon)^{i_0} \geq (r^{-1} - \epsilon)^{n+m^*}$ , and hence (11) holds.

41 The next theorem is an immediate consequence of Theorem 2.  
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**Theorem 3.** Let  $\pi = (\pi_n)$  be a probability distribution and define the tail of  $\pi$  by  $\Pi_n = \sum_{i>n} \pi_i$ . Denote by  $r$  the radius of convergence of the probability generating function  $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$  of  $\pi$ . If  $1 < r < \infty$  and  $\pi(z)$  is meromorphic on  $|z| \leq r$ , then we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \Pi_n = -\log r. \quad (12)$$

**Remark 2.** By Pringsheim's theorem (Theorem 17.13 of Ref.<sup>[5]</sup>), we see that  $z = r$  is a singularity of  $\pi(z)$ .

### 2.3. A Counter Example When the Number of Poles is not Finite

In Theorem 3, we assumed that the number of poles of  $\pi(z)$  on the circle of convergence  $|z| = r$  is finite. We will show below an example of a distribution  $\pi = (\pi_n)$  whose tail does not decay exponentially and  $\pi(z)$  has infinite number of singularities on the circle of convergence.

Let  $\pi = (\pi_n)$  be a probability distribution and put  $\Pi_n = \sum_{i>n} \pi_i$ . For  $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$  and  $\Pi(z) = \sum_{n=0}^{\infty} \Pi_n z^n$ , we have the following lemma.

**Lemma 2.** For  $x > 1$ ,  $\pi(x) < \infty$  and  $\Pi(x) < \infty$  are equivalent.

*Proof.* Since  $\pi_0 = 1 - \Pi_0$ ,  $\pi_n = \Pi_{n-1} - \Pi_n$ ,  $n = 1, 2, \dots$ , we have for  $N \in \mathbb{N}^+$

$$\sum_{n=0}^N \pi_n x^n = 1 + (x-1) \sum_{n=0}^{N-1} \prod_{n=N} x^n - \prod_{n=N} x^N. \quad (13)$$

Hence,  $\Pi(x) < \infty$  implies  $\pi(x) < \infty$ . Conversely, if  $\pi(x) < \infty$ , we have  $\infty > \sum_{n=N+1}^{\infty} \pi_n x^n \geq x^{N+1} \sum_{n=N+1}^{\infty} \pi_n = x^{N+1} \Pi_N$ , which implies  $\Pi(x) < \infty$ .

From Lemma 2, we have the following lemma.

**Lemma 3.** If  $\pi(z)$  has the radius of convergence  $r$  with  $1 < r < \infty$ , then the radius of convergence of  $\Pi(z)$  is also  $r$ , and vice versa.

Now, for an integer  $h \geq 2$ , we define a sequence  $\{v_k\}_{k \geq 0} \subset \mathbb{N}$  by

$$\begin{cases} v_0 = 0, \\ v_{k+1} = v_k + h^{v_k}, \quad k = 0, 1, \dots, \end{cases} \quad (14)$$

and for  $n \in \mathbb{N}$ , we define a function  $\phi(n) \in \mathbb{R}$  by

$$\phi(n) = h^{-v_k}, \quad \text{for } v_k \leq n < v_{k+1}, \quad (15)$$

where  $\mathbb{R}$  is the set of real numbers.  $\phi(n)$  has the following properties;

$$\phi(0) = 1, \quad \phi(n) \geq \phi(n+1), \quad n = 0, 1, \dots, \quad (16)$$

$$\sum_{n=v_k}^{v_{k+1}-1} \phi(n) = h^{-v_k}(v_{k+1} - v_k) = 1, \quad k = 0, 1, \dots, \quad (17)$$

$$\sum_{n=0}^{\infty} \phi(n) = \infty, \quad (18)$$

$$\sum_{n=0}^{\infty} \alpha^{-n} \phi(n) < \infty, \quad \text{for any } \alpha > 1. \quad (19)$$

(16)–(18) are easily seen. (19) can be verified as follows. Since  $v_k \geq k$  holds by induction, we see from (17),

$$\sum_{n=0}^{\infty} \alpha^{-n} \phi(n) = \sum_{k=0}^{\infty} \sum_{n=v_k}^{v_{k+1}-1} \alpha^{-n} \phi(n) \quad (20)$$

$$\leq \sum_{k=0}^{\infty} \alpha^{-v_k} < \infty. \quad (21)$$

We here define a random variable  $T_\alpha \in \mathbb{N}$  for  $\alpha > 1$ , by a probability distribution function

$$P(T_\alpha \leq n) \equiv 1 - \alpha^{-(n+1)} \phi(n+1), \quad n = 0, 1, \dots \quad (22)$$

So, we have

$$P(T_\alpha > n) = \alpha^{-(n+1)} \phi(n+1), \quad n = 0, 1, \dots \quad (23)$$

The probability generating function  $\tau_\alpha(z)$  of  $p_{\alpha,n} \equiv P(T_\alpha = n)$  is defined by

$$\tau_\alpha(z) \equiv \sum_{n=0}^{\infty} p_{\alpha,n} z^n, \quad (24)$$

and the generating function  $T_\alpha(z)$  of  $P(T_\alpha > n)$  by

$$T_\alpha(z) \equiv \sum_{n=0}^{\infty} P(T_\alpha > n) z^n. \quad (25)$$

Then, by (18), (19), we have  $T_\alpha(\alpha) = \infty$  and  $T_\alpha(\alpha') < \infty$  for any  $\alpha' < \alpha$ , so, it follows that  $\alpha$  is the radius of convergence of  $T_\alpha(z)$ . By Cauchy–Hadamard’s theorem, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P(T_\alpha > n) = -\log \alpha. \quad (26)$$

1 On the other hand, for  $n = v_k - 1$ , we have

$$2 \quad P(T_\alpha > v_k - 1) = \alpha^{-v_k} \phi(v_k) \quad (27)$$

$$3 \quad = \alpha^{-v_k} h^{-v_k}, \quad k = 0, 1, \dots, \quad (28)$$

4 hence

$$5 \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \log P(T_\alpha > n) \leq \liminf_{k \rightarrow \infty} \frac{1}{v_k} \log P(T_\alpha > v_k - 1) \quad (29)$$

$$6 \quad = -\log \alpha - \log h \quad (30)$$

$$7 \quad < -\log \alpha. \quad (31)$$

8 From (26), (31), we see that  $\lim_{n \rightarrow \infty} n^{-1} \log P(T_\alpha > n)$  does not exist. Moreover, from  
 9 (30), the difference between  $\limsup$  and  $\liminf$  can be arbitrarily large by taking  $h$   
 10 large. By Lemma 3, the radius of convergence of  $\tau_\alpha(z)$  is equal to  $\alpha$ . The  $\alpha$  satisfies  
 11  $1 < \alpha < \infty$ , however, the tail of the distribution  $(p_{\alpha,n})$  does not decay exponentially.

12 By comparing this example and Theorem 3, we see that the  $\tau_\alpha(z)$  is not mero-  
 13 morphic on  $|z| \leq \alpha$ . In fact, the following theorem describes about the singularities  
 14 of  $\tau_\alpha(z)$  on  $|z| = \alpha$ .

15 **Theorem** (Hadamard's Gap Theorem,<sup>[4]</sup> Theorem 11.7.1). *If a power series  $f(z) =$   
 16  $\sum_{k=0}^{\infty} c_{v_k} z^{v_k}$  satisfies*

$$17 \quad \liminf_{k \rightarrow \infty} \frac{v_{k+1}}{v_k} > 1, \quad (32)$$

18 *then all the points on the circle of convergence are singularities of  $f(z)$ .*

19 This theorem can be applied to our example. Define  $G(z) = (\alpha^{-1}z - 1)T_\alpha(z)$ .  
 20 Since the power series expansion of  $G(z)$  satisfies (32), all the points on the circle  
 21 of convergence  $|z| = \alpha$  are singularities of  $G(z)$ . From the proof of Lemma  
 22 2,  $\tau_\alpha(z) = 1 + (z - 1)T_\alpha(z)$ , and hence, we have  $\tau_\alpha(z) = 1 + (z - 1)G(z)/(\alpha^{-1}z - 1)$ ,  
 23 which implies that all the points on  $|z| = \alpha$  are singularities of  $\tau_\alpha(z)$ .

24 **Remark 3.** Equation (23) is a counter example for Proposition 1 in Ref.<sup>[3]</sup>, which  
 25 overlooks the need for the extra assumption, say, the number of poles on the circle  
 26 of convergence.

### 27 3. EXAMPLE OF M/G/1 TYPE MARKOV CHAIN SUCH THAT 28 THE TAIL OF ITS STATIONARY DISTRIBUTION 29 DOES NOT DECAY EXPONENTIALLY

30 In this section, we show an example of an M/G/1 type Markov chain such that  
 31 the tail of its stationary distribution does not decay exponentially.

1 Consider a Markov chain with the following transition matrix  $P$ .

$$2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad P = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & a_0 & a_1 & a_2 & \dots \\ 0 & 0 & a_0 & a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (33)$$

9 For the probability vectors  $A \equiv (a_0, a_1, \dots)$ ,  $B \equiv (b_0, b_1, \dots)$ , define  $A(z) = \sum_{n=0}^{\infty} a_n z^n$ ,  $B(z) = \sum_{n=0}^{\infty} b_n z^n$ . If  $E(A) \equiv \sum_{n=0}^{\infty} n a_n < 1$  and  $E(B) \equiv \sum_{n=0}^{\infty} n b_n < \infty$ , then there exists the stationary distribution  $\pi = (\pi_n)$  of  $P$ . Define  $\pi(z) = \sum_{n=0}^{\infty} \pi_n z^n$ , then we have

$$10 \quad 11 \quad 12 \quad 13 \quad 14 \quad \pi(z)(z - A(z)) = \pi_0(zB(z) - A(z)). \quad (34)$$

15 Now, let  $A = (1 - q, q, 0, 0, \dots)$  with  $0 < q < 1$ , which represents a Bernoulli process. We have  $A(z) = 1 - q + qz$ . Then define

$$16 \quad 17 \quad 18 \quad 19 \quad B(z) \equiv 1 + z^{-1}(p_{\alpha,0}^{-1}\tau_{\alpha}(z) - 1)(z - A(z)) \quad (35)$$

$$20 \quad 21 \quad = 1 + (1 - q)z^{-1}(p_{\alpha,0}^{-1}\tau_{\alpha}(z) - 1)(z - 1), \quad (36)$$

22 where  $p_{\alpha,0}$  and  $\tau_{\alpha}(z)$  were defined in 2.3. Substituting (36) into (34), we have  $\pi(z) = \tau_{\alpha}(z)$ . We should verify that the coefficients  $B = (b_0, b_1, \dots)$  of  $B(z)$  satisfy  $b_n \geq 0$ ,  $n = 0, 1, \dots$ . In fact, we can show that  $b_n > 0$ ,  $n = 0, 1, \dots$  for  $\alpha \geq 2$ .

23 To prove it, we need the next lemma.

24 **Lemma 4.** For  $\alpha \geq 2$ ,  $p_{\alpha,n}$  decreases monotonically, i.e.,  $p_{\alpha,n} > p_{\alpha,n+1}$ ,  $n = 0, 1, \dots$

25 *Proof.* Since  $p_{\alpha,n} = P(T_{\alpha} > n - 1) - P(T_{\alpha} > n) = \alpha^{-n}\phi(n) - \alpha^{-(n+1)}\phi(n+1)$ ,  $n = 0, 1, \dots$ , we have  $p_{\alpha,n} - p_{\alpha,n+1} = \alpha^{-(n+2)}(\alpha^2\phi(n) - 2\alpha\phi(n+1) + \phi(n+2))$ . Thus, from  $\alpha^2 \geq 2\alpha$ ,  $\phi(n) > 0$ , and  $\phi(n) \geq \phi(n+1)$ , we have  $p_{\alpha,n} > p_{\alpha,n+1}$ .

26 By comparing the coefficients of expansion of the both sides of (36), we have, by Lemma 4, for  $\alpha \geq 2$ ,

$$27 \quad 28 \quad 29 \quad 30 \quad 31 \quad b_0 = 1 - (1 - q)p_{\alpha,0}^{-1}p_{\alpha,1} > 0, \quad (37)$$

$$32 \quad 33 \quad 34 \quad 35 \quad 36 \quad b_n = (1 - q)p_{\alpha,0}^{-1}(p_{\alpha,n} - p_{\alpha,n+1}) > 0, \quad n = 1, 2, \dots \quad (38)$$

37 In summary, we have

38 **Theorem 4.** For  $A = (1 - q, q, 0, 0, \dots)$ ,  $0 < q < 1$ , and  $B = (b_0, b_1, \dots)$  defined by (36) with  $\alpha \geq 2$ , the matrix  $P$  of (33) is irreducible and positive recurrent. The tail of the stationary distribution of  $P$  does not decay exponentially.

39 **Remark 4.** The example in Theorem 4 does not satisfy the condition (i) of Theorem 3.5 in Ref.<sup>[2]</sup>. Is it possible to make an example of an M/G/1 type Markov chain which satisfies the conditions (i), (ii) of Theorem 3.5 in Ref.<sup>[2]</sup> but the tail of its stationary distribution does not decay exponentially?



#### 4. CONCLUSION

In this paper, based on analytic properties of the probability generating function  $\pi(z)$  of a discrete probability distribution  $\pi$ , we proved that if the radius of convergence  $r$  of  $\pi(z)$  satisfies  $1 < r < \infty$  and  $\pi(z)$  is meromorphic on  $|z| \leq r$ , then the tail of  $\pi$  decays exponentially. We displayed a counter example for Proposition 1 in Ref.<sup>[3]</sup>. We also gave an example of an M/G/1 type Markov chain  $P$  such that the tail of the stationary distribution of  $P$  does not decay exponentially.

The examples shown in this paper of probability generating functions whose tail probabilities do not decay exponentially are pathological in the sense that all the points on the circle of convergence are singularities. In almost all the realistic cases,  $\pi(z)$  is expected to be a meromorphic function, and therefore, we can obtain the exponential decay rate of the tail only by calculating the radius of convergence.

#### 5. APPENDIX

##### Proof of Theorem 1

First, we need the following lemma. Denote by  $\mathbb{C}$  the set of complex numbers,  $\mathbb{N}$  the set of natural numbers, and  $\mathbb{N}^+$  the set of positive natural numbers.

**Lemma 5.** *For  $m \in \mathbb{N}^+$ , consider complex numbers  $\beta_i$ ,  $w_i \in \mathbb{C}$ ,  $i = 1, \dots, m$ , with the following properties;*

$$\beta_i \neq 0, \quad |w_i| = 1, \quad w_i \neq w_j \quad \text{for } i \neq j, \quad i, j = 1, \dots, m. \quad (39)$$

Define  $t_n \equiv \sum_{i=1}^m \beta_i w_i^n$  for  $n \in \mathbb{N}$ . Then there exists an  $\epsilon > 0$  such that for any  $N \in \mathbb{N}$  there is at least one  $n$  among consecutive  $m$  natural numbers  $N, N+1, \dots, N+m-1$  with  $|t_n| > \epsilon$ .

*Proof.* Suppose that the claim were not true. Then for any  $\epsilon > 0$  there must exist an  $N \in \mathbb{N}$  such that  $|t_n| \leq \epsilon$  holds for  $n = N, N+1, \dots, N+m-1$ . Consider the matrix representation

$$\begin{pmatrix} t_N \\ t_{N+1} \\ \vdots \\ t_{N+m-1} \end{pmatrix} = \begin{pmatrix} w_1^N & w_2^N & \dots & w_m^N \\ w_1^{N+1} & w_2^{N+1} & \dots & w_m^{N+1} \\ \dots & \dots & \dots & \dots \\ w_1^{N+m-1} & w_2^{N+m-1} & \dots & w_m^{N+m-1} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{pmatrix}. \quad (40)$$

Let  $V_N$  be the matrix of the right hand side of (40). We have

$$\begin{aligned} \det V_N &= w_1^N w_2^N \dots w_m^N \times \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ w_1 & w_2 & \dots & w_m \\ \dots & \dots & \dots & \dots \\ w_1^{m-1} & w_2^{m-1} & \dots & w_m^{m-1} \end{pmatrix} \\ &= w_1^N w_2^N \dots w_m^N (-1)^{m(m-1)/2} \prod_{i < j} (w_i - w_j), \end{aligned}$$

(Vandermonde's determinant) and hence

$$|\det V_N| = \prod_{i < j} |w_i - w_j| \equiv D > 0. \quad (41)$$

Denoting by  $\Delta_{ij}$  the adjoint of  $V_N$ , we have

$$|\Delta_{ij}| \leq (m-1)!, \quad i, j = 1, \dots, m, \quad (42)$$

because  $\Delta_{ij}$  is the sum of  $(m-1)!$  terms of absolute value 1. From (40), we have

$$\beta_i = \frac{1}{\det V_N} \sum_{j=1}^m \Delta_{ij} t_{N+j-1}, \quad i = 1, \dots, m, \quad (43)$$

thus,

$$|\beta_i| \leq \frac{1}{D} \sum_{j=1}^m |\Delta_{ij}| |t_{N+j-1}| \quad (44)$$

$$\leq \frac{m!}{D} \epsilon, \quad i = 1, \dots, m. \quad (45)$$

Since  $\epsilon > 0$  is arbitrary, from (45), we have  $\beta_i = 0$ , but this contradicts the assumption of the lemma.

Next, we prove the following proposition which corresponds to the case  $r = 1$  of Theorem 1.

**Proposition 1.** *Assume that the radius of convergence of  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  is equal to 1, and  $f(z)$  is meromorphic on  $|z| \leq 1$ . Let  $m$  be the number of poles on  $|z| = 1$ . Then, there exist  $m^* \in \mathbb{N}^+$ ,  $1 \leq m^* \leq m$ , and a sequence  $\{n_k\}_{k \geq 0} \subset \mathbb{N}$  which satisfy*

$$\lim_{k \rightarrow \infty} |c_{n_k}|^{1/n_k} = 1, \quad (46)$$

$$km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \dots \quad (47)$$

*Proof* (cf. [8], p.152, problem 242). Let  $\zeta_1, \dots, \zeta_m$  be the poles of  $f(z)$  on the circle of convergence  $|z| = 1$ , and  $s_i$  the order of the pole  $\zeta_i$ ,  $i = 1, \dots, m$ . Let  $p_i(z)$  be the principal part of  $f(z)$  at  $\zeta_i$  with

$$p_i(z) = \frac{1}{(\zeta_i - z)^{s_i}} \sum_{l=0}^{s_i-1} a_{il} z^l, \quad i = 1, \dots, m. \quad (48)$$

Since  $\sum_{l=0}^{s_i-1} a_{il} z^l$  does not have a factor  $\zeta_i - z$ , we see  $\sum_{l=0}^{s_i-1} a_{il} \zeta_i^l \neq 0$ ,  $i = 1, \dots, m$ . Put  $g(z) \equiv f(z) - \sum_{i=1}^m p_i(z)$ , then  $g(z)$  is holomorphic on  $|z| \leq 1$ . Let  $g(z) = \sum_{n=0}^{\infty} d_n z^n$  be the power series expansion of  $g(z)$ , then by Cauchy-Hadamard's theorem, we have  $\limsup_{n \rightarrow \infty} |d_n|^{1/n} < 1$ . Let  $p_i(z) = \sum_{n=0}^{\infty} e_{in} z^n$  be the expansion of  $p_i(z)$ . Using the expansion

$$\frac{1}{(\zeta - z)^s} = \frac{1}{\zeta^s} \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} \zeta^{-n} z^n, \quad (49)$$

1 we have

$$2 \quad e_{in} = \sum_{l=0}^{s_i-1} \binom{n-l+s_i-1}{s_i-1} \zeta_i^{\zeta_i^{-s_i-n+l}} a_{il}, \quad (50)$$

3 for  $n \geq s_i - 1$ ,  $i = 1, \dots, m$ . Then by  $f(z) = \sum_{i=1}^m p_i(z) + g(z)$ , we have

$$4 \quad c_n = \sum_{i=1}^m \sum_{l=0}^{s_i-1} \binom{n-l+s_i-1}{s_i-1} \zeta_i^{\zeta_i^{-s_i-n+l}} a_{il} + d_n, \quad (51)$$

5 for  $n \geq \max_{1 \leq i \leq m} (s_i - 1)$ .

6 Since  $\binom{n-l+s_i-1}{s_i-1} = n^{s_i-1} (1 + O(n^{-1}))$ , we have, from (50),

$$7 \quad e_{in} = n^{s_i-1} \left( \zeta_i^{\zeta_i^{-s_i-n}} \sum_{l=0}^{s_i-1} \zeta_i^l a_{il} \right) (1 + O(n^{-1})). \quad (52)$$

8 Let  $S \equiv \max_{1 \leq i \leq m} s_i$ . Without loss of generality, we can assume that  $i = 1, \dots, m^* (\leq m)$  attain the maximum. From (51), we have

$$9 \quad c_n = n^{S-1} \left( \sum_{i=1}^{m^*} \zeta_i^{\zeta_i^{-S-n}} \sum_{l=0}^{S-1} a_{il} \zeta_i^l \right) (1 + O(n^{-1})), \quad (53)$$

10 and then

$$11 \quad |c_n|^{1/n} = (n^{1/n})^{S-1} \left| \sum_{i=1}^{m^*} \zeta_i^{\zeta_i^{-S-n}} \sum_{l=0}^{S-1} a_{il} \zeta_i^l \right|^{1/n} |1 + O(n^{-1})|^{1/n}. \quad (54)$$

12 In the right hand side of (54), we see that  $\lim_{n \rightarrow \infty} (n^{1/n})^{S-1} = 1$ ,  $\lim_{n \rightarrow \infty} |1 + O(n^{-1})|^{1/n} = 1$ . Meanwhile, about the second factor of (54), writing

$$13 \quad w_i \equiv \zeta_i^{-1}, \quad \beta_i \equiv \zeta_i^{-S} \sum_{l=0}^{S-1} a_{il} \zeta_i^l, \quad i = 1, \dots, m^*, \quad (55)$$

14 we have

$$15 \quad \beta_i \neq 0, \quad |w_i| = 1, \quad w_i \neq w_j \quad \text{for } i \neq j, \quad i, j = 1, \dots, m^*. \quad (56)$$

16 Then by Lemma 5, there exist  $\epsilon > 0$  and a sequence  $\{n_k\} \subset \mathbb{N}$ , such that

$$17 \quad \left| \sum_{i=1}^{m^*} \zeta_i^{-S-n_k} \sum_{l=0}^{S-1} a_{il} \zeta_i^l \right| > \epsilon, \quad (57)$$

$$18 \quad km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \dots$$

19 In summary, we have

$$20 \quad \lim_{k \rightarrow \infty} |c_{n_k}|^{1/n_k} = 1, \quad (58)$$

$$21 \quad km^* \leq n_k < (k+1)m^*, \quad k = 0, 1, \dots \quad (59)$$

1 *Proof of Theorem 1.* Apply Proposition 1 to the power series  $\sum_{n=0}^{\infty} c_n r^n z^n$  with the  
2 radius of convergence 1.  
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#### 5 REFERENCES

- 6
- 7 1. Duffield, N.G. Exponential bounds for queues with Markovian arrivals.  
8 Queueing Systems **1994**, *17*, 413–430.
- 9 2. Falkenberg, E. On the asymptotic behaviour of the stationary distribution of  
10 Markov chain of M/G/1-type. Stochastic Models **1994**, *10*, 75–98.
- 11 3. Glynn, P.W.; Whitt, W. Logarithmic asymptotics for steady-state tail probabilities  
12 in a single-server queue. Adv. Appl. Prob. **1994**, *31A*, 131–156.
- 13 4. Hille, E. *Analytic Function Theory*; Chelsea Publishing Company: New York,  
14 1977; Vol. II.
- 15 5. Markushevich, A.I. *Theory of Functions of a Complex Variable*; Chelsea  
16 Publishing Company: New York, 1985.
- 17 6. Neuts, M.F. *Structured Stochastic Matrix of M/G/1 Type and Their Applications*;  
18 Marcel Dekker: New York, 1989.
- 19 7. Neuts, M.F. *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic*  
20 *Approach*; Johns Hopkins University Press: Baltimore, 1981.
- 21 8. Pólya, G.; Szegő, G. *Problems and Theorems in Analysis*; Springer Verlag, 1972;  
22 Vol. I.
- 23 9. Takahashi, Y. Asymptotic exponentiality of the tail of the waiting-time  
24 distribution in a PH/PH/c queue. Adv. Appl. Prob. **1981**, *13*, 619–630.
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